MAC 1105

Module 10
Higher-Degree Polynomial Functions

Learning Objectives

Upon completing this module, you should be able to

1. Identify intervals where a function is increasing or decreasing.
2. Find extrema of a function.
3. Identify symmetry in a graph of a function.
4. Determine if a function is odd, even, or neither.
5. Recognize and graph polynomial functions.

Higher-Degree Polynomial Functions

There are two major topics in this module:

- Nonlinear Functions and Their Graphs
- Polynomial Functions and Models

Let's get started by looking at some polynomial functions.

Polynomial Functions

Polynomial functions are frequently used to approximate data.

POLYNOMIAL FUNCTION

A **polynomial function f of degree n in the variable x** can be represented by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where each coefficient a_k is a real number, $a_n \neq 0$, and n is a nonnegative integer. The **leading coefficient** is a_n and the **degree** is n.

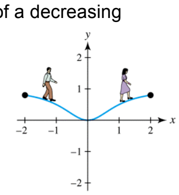
A polynomial function of degree 2 or higher is a nonlinear function.

Is the Function Increasing or Decreasing?

The concept of increasing and decreasing relate to whether the graph of a function rises or falls.

- Moving from left to right along a graph of an increasing function would be uphill.
- Moving from left to right along a graph of a decreasing function would be downhill.

We speak of a function f increasing or decreasing *over an interval of its domain*.

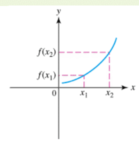


Is the Function Increasing or Decreasing? (Cont.)

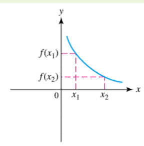
INCREASING AND DECREASING FUNCTIONS

Suppose that a function f is defined over an interval I on the number line. If x_1 and x_2 are in I,

(a) f **increases** on I if, whenever $x_1 < x_2$, $f(x_1) < f(x_2)$;
(b) f **decreases** on I if, whenever $x_1 < x_2$, $f(x_1) > f(x_2)$.



When $x_1 < x_2$, $f(x_1) < f(x_2)$, f is increasing.



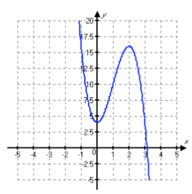
When $x_1 < x_2$, $f(x_1) > f(x_2)$, f is decreasing.

Example

Use the graph of $f(x) = -3x^3 + 9x^2 + 4$ shown below and interval notation to identify where f is increasing or decreasing.

Solution

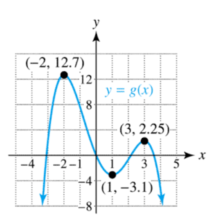
Moving from left to right on the graph of f, the y-values decreases until $x = 0$, increases until $x = 2$, and decreases thereafter. Thus, in interval notation f is decreasing on $(-\infty, 0] \cup [2, \infty)$.



Extrema of Nonlinear Functions

Graphs of polynomial functions often have "hills" or "valleys".

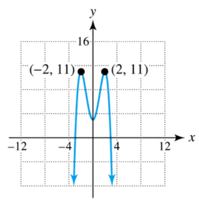
- The "highest hill" on the graph is located at (–2, 12.7). This is the ***absolute maximum*** of *g*.
- There is a smaller peak located at the point (3, 2.25). This is called the ***local maximum***.



Extrema of Nonlinear Functions (Cont.)

Maximum and minimum values that are either absolute or local are called **extrema.**

- A function may have several local extrema, but at most one absolute maximum and one absolute minimum.
- It is possible for a function to assume an absolute extremum at two values of x.
- The absolute maximum is 11. It is a local maximum as well, because near $x = -2$ and $x = 2$ it is the largest y-value.



Absolute and Local Extrema

ABSOLUTE AND LOCAL EXTREMA

Let c be in the domain of f.

$f(c)$ is an **absolute (global) maximum** if $f(c) \geq f(x)$ *for all* x in the domain of f.

$f(c)$ is an **absolute (global) minimum** if $f(c) \leq f(x)$ *for all* x in the domain of f.

$f(c)$ is a **local (relative) maximum** if $f(c) \geq f(x)$ when x is *near* c.

$f(c)$ is a **local (relative) minimum** if $f(c) \leq f(x)$ when x is *near* c.

The absolute maximum is the maximum y-value on the graph $y = f(x)$.

Example

The monthly average ocean temperature in degrees Fahrenheit at Bermuda can be modeled by

$$f(x) - 0.0215x^4 - 0.648x^3 + 6.03x^2 - 17.1x + 76.4,$$

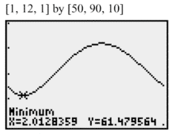
where $x = 1$ corresponds to January and $x = 12$ to December. The domain of f is $D = \{x | 1 \leq x \leq 12\}$.

(Source: J. Williams, *The Weather Almanac* 1995.)

- Graph f in [1, 12, 1] by [50, 90, 10].
- Estimate the absolute extrema. Interpret the results.

Solution

[1, 12, 1] by [50, 90, 10]

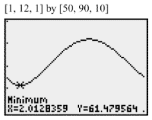


Example (Cont.)

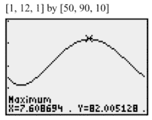
b) Many graphing calculators have the capability to find maximum and minimum *y*-values.

- An absolute minimum of about 61.5 corresponds to the point (2.01, 61.5). This means the monthly average ocean temperature is coldest during February, when it reaches $61.5^{\circ}F$.
- An absolute maximum of about 82 corresponds to the point (7.61, 82.0), meaning that the warmest average ocean temperature occurs in August when it reaches a maximum of $82^{\circ}F$.

[1, 12, 1] by [50, 90, 10]



[1, 12, 1] by [50, 90, 10]

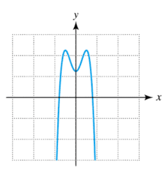


What is an Even Function?

EVEN FUNCTION

A function f is an **even function** if $f(-x) = f(x)$ for every x in its domain. The graph of an even function is symmetric with respect to the y-axis.

If a graph was folded along the *y*-axis, and the right and left sides would match, the graph would be *symmetric with respect to the y-axis*. A function whose graph satisfies this characteristic is called an **even function**.

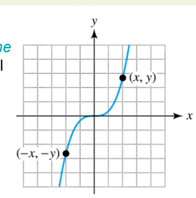


What is an Odd Function?

ODD FUNCTION

A function f is an **odd function** if $f(-x) = -f(x)$ for every x in its domain. The graph of an odd function is symmetric with respect to the origin.

When the symmetry occurs in respect *to the origin*. If the graph could rotate, the original graph would reappear after half a turn (clockwise or counter-clockwise.) This represents an **odd function**.



Is the Function Even or Odd?

Identify whether the function is even or odd.

$$f(x) = 6x^3 - 9x$$

Solution

Since *f* is a polynomial containing only odd powers of *x*, it is an *odd function*. This also can be shown symbolically as follows.

$$\begin{aligned} f(-x) &= 6(-x)^3 - 9(-x) \\ &= -6x^3 + 9x \\ &= -(6x^3 - 9x) \\ &= -f(x) \end{aligned}$$

Turning Points of a Polynomial Function

A polynomial function *f* of degree *n* can be expressed as

$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$, where each coefficient a_k is a real number, $a_n \neq 0$, and *n* is a nonnegative integer.

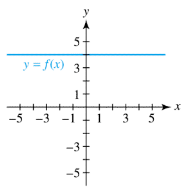
A turning point occurs whenever the graph of a polynomial function changes from increasing to decreasing or from decreasing to increasing.

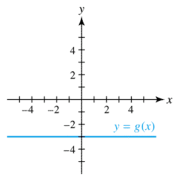
Turning points are associated with "hills" or "valleys" on a graph.

What are the characteristics of different types of polynomial functions?

Constant Polynomial Functions

- No *x*-intercepts.
- No turning points.





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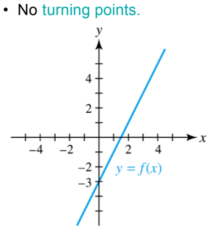
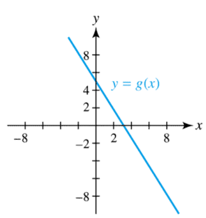
Linear Polynomial Functions

- Degree 1.
- One *x*-intercepts.
- No turning points.





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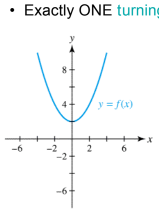
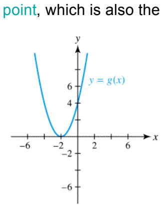
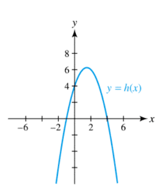
Quadratic Polynomial Functions

- Degree 2 - parabola that opens up or down.
- Zero, one or two *x*-intercepts.
- Exactly ONE turning point, which is also the VERTEX.







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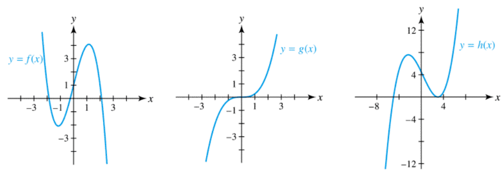
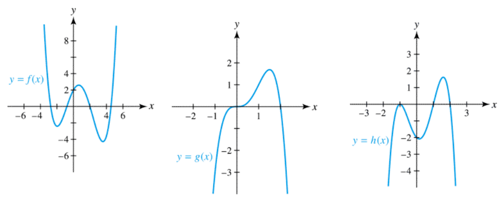
Cubic Polynomial Functions

- Degree 3.
- Up to three x-intercepts.
- Up to two turning points.



Quartic Polynomial Functions

- Degree 4.
- Up to four x-intercepts.
- Up to three turning points.



End Behavior of Polynomial Functions

END BEHAVIOR OF POLYNOMIAL FUNCTIONS

Let f be a polynomial function with leading coefficient a and degree n.

1. $n \geq 2$ is even.
$a > 0$ implies the graph of f rises both to the left and to the right. That is, $f(x) \to \infty$ as $x \to \pm\infty$.
$a < 0$ implies the graph of f falls both to the left and to the right. That is, $f(x) \to -\infty$ as $x \to \pm\infty$.

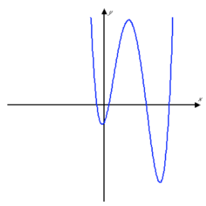
2. $n \geq 1$ is odd.
$a > 0$ implies the graph of f falls to the left and rises to the right. That is, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.
$a < 0$ implies the graph of f rises to the left and falls to the right. That is, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

Main Characteristics of Polynomial Functions

Degree, *x*-intercepts, and turning points

The graph of a polynomial function of degree $n \geq 1$ has at most n x-intercepts and at most $n - 1$ turning points.

Example



Use the graph of the polynomial function shown.

a) How many turning points and x-intercepts are there?

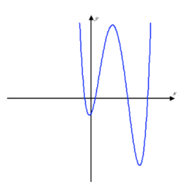
b) Is the leading coefficient a positive or negative? Is the degree odd or even?

c) Determine the minimum degree of f.

Solution

a) There are three turning points corresponding to the one "hill" and two "valleys". There appear to be 4 x-intercepts.

Example (Cont.)



b) Is the leading coefficient a positive or negative? Is the degree odd or even?

The left side and the right side rise.

Therefore, $a > 0$ and the polynomial function has even degree.

c) Determine the minimum degree of f.

The graph has three turning points. A polynomial of degree n can have at most $n - 1$ turning points.

Therefore, f must be at least degree 4.

Let's Practice One More Time

Let $f(x) = 3x^4 + 5x^3 - 2x^2$.

a) Give the degree and leading coefficient.

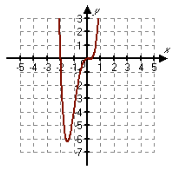
b) State the end behavior of the graph of *f*.

Solution

a) The term with the highest degree is $3x^4$ so the degree is 4 and the leading coefficient is 3.

b) The degree is even and the leading coefficient is positive. Therefore the graph of *f* rises to the left and right. More formally,

$$f(x) \to \infty \text{ as } x \to \pm\infty$$



What have we learned?

We have learned to:

1. Identify intervals where a function is increasing or decreasing.
2. Find extrema of a function.
3. Identify symmetry in a graph of a function.
4. Determine if a function is odd, even, or neither.
5. Recognize and graph polynomial functions.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition