MAC 1105

Module 11

Solution of Polynomial Equations

Learning Objective

Upon completing this module, you should be able to:

1. Solve polynomial equations using factoring.
2. Solve polynomial equations using factoring by grouping.
3. Solve polynomial equations using the root method.
4. Find factors, zeros, x-intercepts, and solutions.

Quick Review of Polynomial Division

$$
\begin{array}{r}
x^2 + 5x + 10 \\
x - 3 \overline{\smash{\big)} x^3 + 2x^2 - 5x - 6} \\
\underline{x^3 - 3x^2} \qquad\qquad \\
5x^2 - 5x \qquad \\
\underline{5x^2 - 15x} \qquad \\
10x - 6 \\
\underline{10x - 30} \\
24
\end{array}
$$

The quotient is $x^2 + 5x + 10$ with a remainder of 24.

Check:

$$(x-3)(x^2+5x+10)+24 = x(x^2+5x+10) - 3(x^2+5x+10) + 24$$

$$= x^3 + 5x^2 + 10x - 3x^2 - 15x - 30 + 24$$

$$= x^3 + 2x^2 - 5x - 6$$

Division Algorithm for Polynomials

DIVISION ALGORITHM FOR POLYNOMIALS

Let $f(x)$ and $d(x)$ be two polynomials with the degree of $d(x)$ greater than zero and less than the degree of $f(x)$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)} \cdot \underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)},$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. The polynomial $r(x)$ is called the remainder.

Remainder Theorem and Factor Theorem

REMAINDER THEOREM

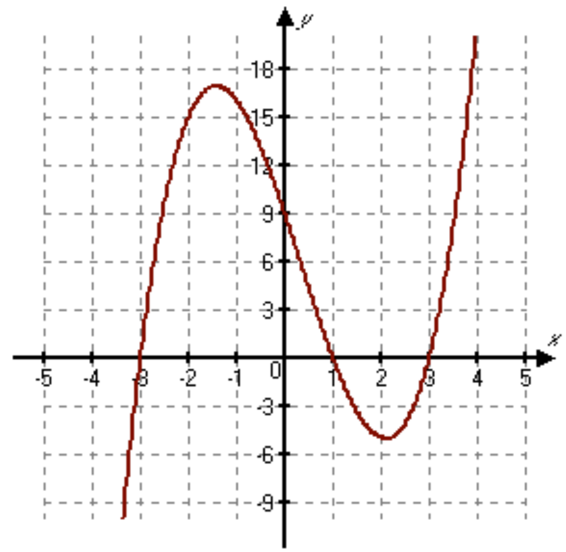
If a polynomial $f(x)$ is divided by $x - k$, the remainder is $f(k)$.

FACTOR THEOREM

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Example

Use the graph of $f(x) = x^3 - x^2 - 9x + 9$ and the factor theorem to list the factors of $f(x)$.



Solution

The graph shows that the zeros or x-intercepts of f are -3, 1and 3.

Since $f(-3) = 0$, the factor theorem states that $(x + 3)$ is a factor, and

$f(1) = 0$ implies that $(x - 1)$ is a factor, and

$f(3) = 0$ implies $(x - 3)$ is a factor.

Thus the factors are $(x + 3)(x - 1)$, and $(x - 3)$.

Complete Factored Form

COMPLETE FACTORED FORM

Suppose a polynomial

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

has n real zeros $c_1, c_2, c_3, \ldots, c_n$, where distinct zeros are listed as many times as their multiplicities. Then $f(x)$ can be written in **complete factored form** as

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n).$$

Example

Find all real solutions to the equation

$4x^4 - 17x^2 - 50 = 0.$

Solution

The expression can be factored similar to a quadratic equation.

$$4x^4 - 17x^2 - 50 = 0$$

$$(4x^2 - 25)(x^2 + 2) = 0$$

$$4x^2 - 25 = 0 \quad \text{or} \quad x^2 + 2 = 0$$

$$4x^2 = 25 \qquad x^2 = -2$$

$$x^2 = \frac{25}{4} \qquad x^2 = -2$$

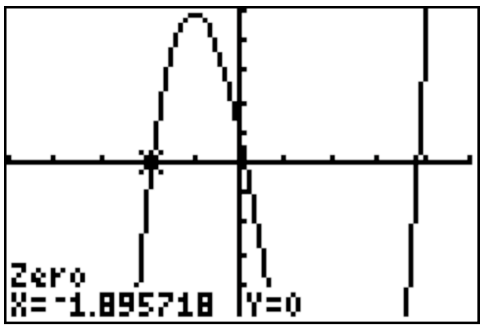
$$x = \pm\frac{5}{2} \qquad x^2 = -2$$

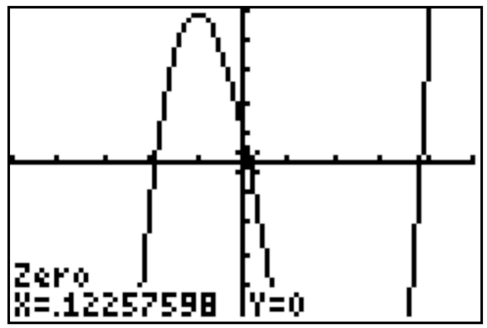
The only solutions are $\pm\frac{5}{2}$ since the equation $x^2 = -2$ has no real solutions.

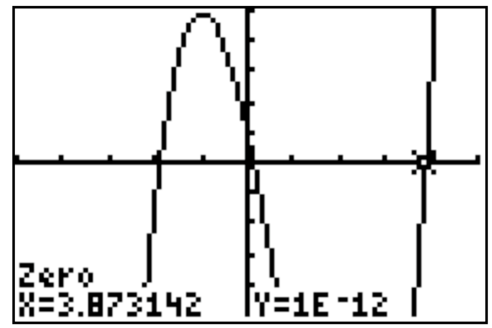
Another Example

Solve the equation $x3 – 2.1x2 – 7.1x + 0.9 = 0$ graphically. Round any solutions to the nearest hundredth.

Solution







Since there are three *x*-intercepts the equation has three real solutions.

$x \approx .012, –1.89,$ and 3.87

Quick Review on Complex Number

A complex number can be written in **standard form** as $a + bi$, where a and b are real numbers. The **real part** is a and the **imaginary part** is b.

PROPERTIES OF THE IMAGINARY UNIT *i*

$$i = \sqrt{-1}, \qquad i^2 = -1$$

THE EXPRESSION $\sqrt{-a}$

If $a > 0$, then $\sqrt{-a} = i\sqrt{a}$.

Examples

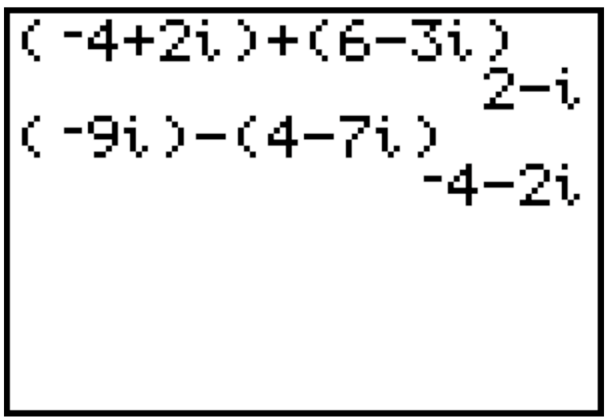
Write each expression in standard form.
Support your results using a calculator.

$(-4 + 2i) + (6 - 3i)$ b) $(-9i) - (4 - 7i)$

Solution

a) $(-4 + 2i) + (6 - 3i) = -4 + 6 + 2i - 3i = 2 - i$

b) $(-9i) - (4 - 7i) = -4 - 9i + 7i = -4 - 2i$



Examples

Write each expression in standard form. Support your results using a calculator.

a) $(-2 + 5i)^2$

b) $\dfrac{16}{3+i}$

Solution

a) $(-2 + 5i)^2 = (-2 + 5i)(-2 + 5i)$

$= 4 - 10i - 10i + 25i^2$

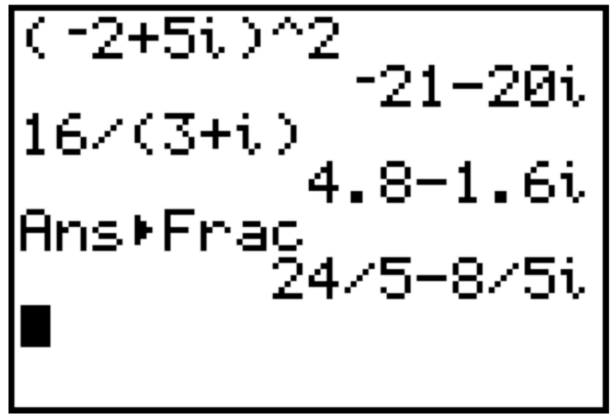
$= 4 - 20i + 25(-1)$

$= -21 - 20i$

b) $$\frac{16}{3+i} = \frac{16}{3+i} \times \frac{16}{3-i}$$

$$= \frac{48-16i}{9-i^2}$$

$$= \frac{48-16i}{10} = \frac{24}{5} \pm \frac{8}{5}i$$



Quadratic Equations with Complex Solutions

We can use the quadratic formula to solve quadratic equations, even if the discriminant is negative.

However, there are no real solutions, and the graph does not intersect the x-axis.

The solutions can be expressed as imaginary numbers.

Example

Solve the quadratic equation $4x^2 - 12x = -11$.

Solution

Rewrite the equation: $4x^2 - 12x + 11 = 0$

$a = 4$, $b = -12$, $c = 11$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{(12)^2 - 4(4)(11)}}{2(4)}$$

$$= \frac{12 \pm \sqrt{-32}}{8}$$

$$= \frac{12 \pm 4i\sqrt{2}}{8} = \frac{3}{2} \pm \frac{i\sqrt{2}}{2}$$

Example

Represent a polynomial of degree 4 with leading coefficient 3 and zeros of -2, 4, i and $-i$ in complete factored form and expanded form.

Solution

Let $a_n = 3$, $c_1 = -2$, $c_2 = 4$, $c_3 = i$, and $c_4 = -i$.

$f(x) = 3(x + 2)(x - 4)(x - i)(x + i)$

Expanded: $3(x + 2)(x - 4)(x - i)(x + i)$

$= 3(x + 2)(x - 4)(x^2 + 1)$

$= 3(x + 2)(x^3 - 4x^2 + x - 4)$

$= 3(x^4 - 2x^3 - 7x^2 - 2x - 8)$

$= 3x^4 - 6x^3 - 21x^2 - 6x - 24$

What is the Conjugate Zeros Theorem?

Conjugate Zeros Theorem

If a polynomial $f(x)$ has only real coefficients and if $a + bi$ is a zero of $f(x)$, then the conjugate $a - bi$ is also a zero of $f(x)$.

Example of Applying the Conjugate Zeros Theorem

Find the zeros of $f(x) = x^4 + 5x^2 + 4$ given one zero is $-i$.

Solution

By the conjugate zeros theorem it follows that i must be a zero of $f(x)$.

$(x + i)$ and $(x - i)$ are factors

$(x + i)(x - i) = x^2 + 1$, using long division we can find another quadratic factor of $f(x)$.

$$
\begin{array}{r}
x^2 + 4 \\
x^2 + 0x + 1 \overline{\big)\, x^4 + 0x^3 + 5x^2 + 0x + 4} \\
\underline{x^4 + 0x^3 + x^2} \\
4x^2 + 0x + 4 \\
\underline{4x^2 + 0x + 4} \\
0
\end{array}
$$

The solution is

$x^4 + 5x^2 + 4 = (x^2 + 4)(x^2 + 1)$

Another Example

Solve $x^3 = 2x^2 - 5x + 10$.

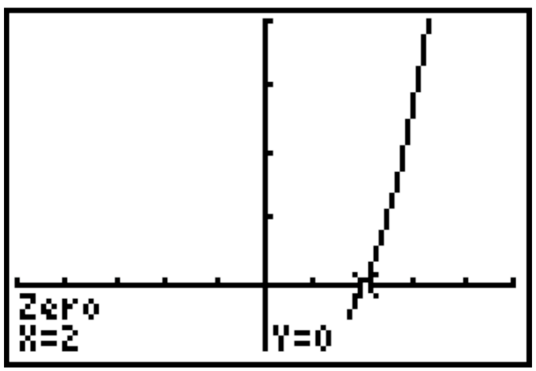
Solution

Rewrite the equation: $f(x) = 0$, where

$f(x) = x^3 - 2x^2 + 5x - 10$

We can use factoring by grouping or graphing to find one real zero.

$f(x) = (x^3 - 2x^2) + (5x - 10) = x^2(x - 2) + 5(x - 2) = (x^2 + 5)(x - 2)$



$$x - 2 = 0 \quad \text{or} \quad x^2 + 5 = 0$$
$$x = 2 \quad \text{or} \quad x^2 = -5$$
$$x = 2 \quad \text{or} \quad x = \pm i\sqrt{5}$$

The graph shows a zero at 2.
So, $x - 2$ is a factor.

How to Solve Equations Involving Rational Exponents?

Example Solve $4x^{3/2} - 6 = 6$. Approximate the answer to the nearest hundredth, and give graphical support.

Solutions

Symbolic Solution | *Graphical Solution*

$4x^{3/2} - 6 = 6$

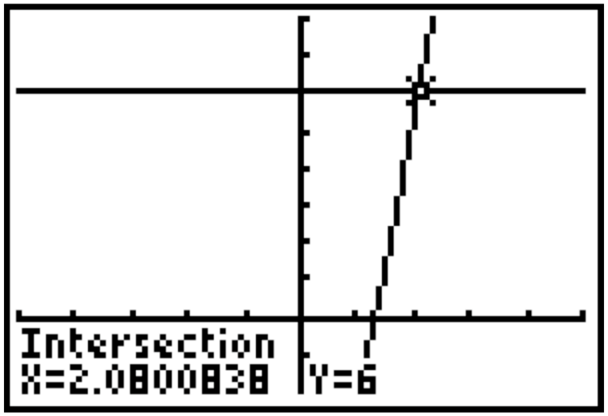
$4x^{3/2} = 12$

$(x^{3/2})^2 = 3^2$

$x^3 = 9$

$x = 9^{1/3}$

$x = 2.08$



Square Root Property

SQUARE ROOT PROPERTY

Let k be a nonnegative number. Then the solutions to the equation

$$x^2 = k$$

are given by $x = \pm\sqrt{k}$.

Example:

$$-16t^2 + 68 = 0$$

$$-16t^2 = -68$$

$$t^2 = \frac{-68}{-16}$$

$$t^2 = \frac{17}{4}$$

$$t = \pm\sqrt{\frac{17}{4}}$$

$$t \approx \pm 2.1$$

Note: We can solve quadratic equations of the form $x^2 = C$ by taking the square root of both sides. We get both the positive and negative roots.

In the same manner, we can solve $x^3 = C$ by taking the cube root of both sides. However, there is only one real cube root of a number.

In general, there is only one real *n*th root if n is odd, and there are zero or two real roots if n is even.

How to Solve an Equation Involving Square Root?

When solving equations that contain square roots, it is common to square each side of an equation.

Example Solve $\sqrt{3x-2} = x-2.$

Solution

$$\sqrt{3x-2} = x-2$$

$$\left(\sqrt{3x-2}\right)^2 = (x-2)^2$$

$$3x-2 = x^2-4x+4$$

$$x^2-7x+6=0$$

$$(x-1)(x-6)=0$$

$$x=1 \text{ or } x=6$$

How to Solve an Equation Involving Square Root? (Cont.)

Check

$$\sqrt{3(1)-2} = 1-2 \qquad\qquad \sqrt{3(6)-2} = 6-2$$

$$1 \neq -1 \qquad\qquad 4 = 4$$

Substituting these values in the original equation shows that the value of 1 is an **extraneous solution** because it does not satisfy the given equation.

Therefore, the only solution is 6.

Example of Solving an Equation Involving Cube Roots?

Some equations may contain a cube root.

Solve $\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}.$

Solution

$$\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}$$

$$\left(\sqrt[3]{4x^2 - 4x + 1}\right)^3 = \left(\sqrt[3]{x}\right)^3$$

$$4x^2 - 5x + 1 = 0$$

$$(4x - 1)(x - 1) = 0$$

$$x = \frac{1}{4} \quad \text{or} \quad x = 1$$

Both solutions check, so the solution set is $\left\{\frac{1}{4},\ 1\right\}.$

What have we learned?

We have learned to:

1. Solve polynomial equations using factoring.
2. Solve polynomial equations using factoring by grouping.
3. Solve polynomial equations using the root method.
4. Find factors, zeros, x-intercepts, and solutions.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition