MAC 1105

Module 12

Rational Functions and Rational Equations

Learning Objective

Upon completing this module, you should be able to:

1. Identify a rational function and state its domain.
2. Find and interpret vertical asymptotes.
3. Find and interpret horizontal asymptotes.
4. Solve rational equations.

What is a Rational Function?

RATIONAL FUNCTION

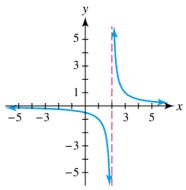
A function f represented by $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is a **rational function**.

A rational function is a nonlinear function. The domain of a rational function includes all real numbers except the zeros of the denominator *q(x)*.

What is a Vertical Asymptote?

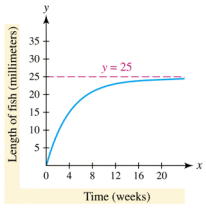
VERTICAL ASYMPTOTE

The line $x = k$ is a **vertical asymptote** of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$, as x approaches k from either the left or the right.



In this graph, the line x = 2 is a vertical asymptote.

What is a Horizontal Asymptote?



In this graph, the line y = 25 is a horizontal asymptote.

HORIZONTAL ASYMPTOTE

The line $y = b$ is a **horizontal asymptote** of the graph of f, if $f(x) \to b$ as x approaches either ∞ or $-\infty$.

Example

Use the graph of $f(x) = \frac{1}{x^2}$ to sketch the graph of $g(x) = \frac{1}{(x+1)^2} - 2$.

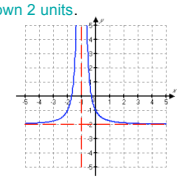
Include all asymptotes in your graph. Write $g(x)$ in terms of $f(x)$.

Solution

$g(x)$ is a translation of $f(x)$ left one unit and down 2 units.

The vertical asymptote is $x = -1$
The horizontal asymptote is $y = -2$

$g(x) = f(x + 1) - 2$



How to Find Vertical and Horizontal Asymptotes?

FINDING VERTICAL AND HORIZONTAL ASYMPTOTES

Let f be a rational function given by $f(x) = \frac{p(x)}{q(x)}$ written in lowest terms.

Vertical Asymptote

To find a vertical asymptote, set the denominator, $q(x)$, equal to 0 and solve. If k is a zero of $q(x)$, then $x = k$ is a vertical asymptote. *Caution:* If k is a zero of both $q(x)$ *and* $p(x)$, then $f(x)$ is *not* written in lowest terms, and $x - k$ is a common factor.

Horizontal Asymptote

(a) If the degree of the numerator is less than the degree of the denominator, then $y = 0$ (the x-axis) is a horizontal asymptote.

(b) If the degree of the numerator equals the degree of the denominator, then $y = \frac{a}{b}$ is a horizontal asymptote, where a is the leading coefficient of the numerator, and b is the leading coefficient of the denominator.

(c) If the degree of the numerator is greater than the degree of the denominator, then there are no horizontal asymptotes.

Examples

For each rational function, determine any horizontal or vertical asymptotes.

a) $f(x) = \frac{2x+6}{4x-8}$ b) $f(x) = \frac{x-1}{x^2-9}$ c) $f(x) = \frac{x^2-4}{x-2}$

Solution

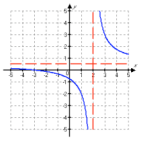
To identify horizontal asymptote, look at the leading coefficient of the highest power term in both numerator and denominator.

- Horizontal Asymptote: If the Degree of numerator equals the degree of the denominator,

  $y = a/b$ is asymptote, so $y = 2/4 = 1/2$

To identify vertical asymptote, set the denominator to 0.

Vertical Asymptote: $4x - 8 = 0$, $x = 2$



Examples (Cont.)

For each rational function, determine any horizontal or vertical asymptotes.

a) $f(x) = \frac{2x+6}{4x-8}$ b) $f(x) = \frac{x-1}{x^2-9}$ c) $f(x) = \frac{x^2-4}{x-2}$

Solution (Cont.)

b) Horizontal Asymptote:

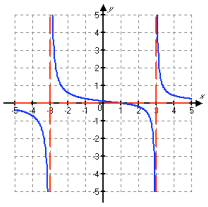
Degree: numerator < denominator

$y = 0$ is the horizontal asymptote.

Vertical Asymptote:

$x^2 - 9 = 0$

$x = \pm 3$ are the vertical asymptotes.



Examples (Cont.)

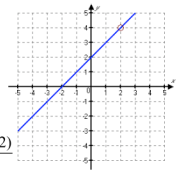
For each rational function, determine any horizontal or vertical asymptotes.

a) $f(x) = \dfrac{2x+6}{4x-8}$ b) $f(x) = \dfrac{x-1}{x^2-9}$ c) $f(x) = \dfrac{x^2-4}{x-2}$

Solution (Cont.)

c) Horizontal Asymptote:
Degree: numerator > denominator
no horizontal asymptotes
Vertical Asymptote:
no vertical asymptotes
(There will be a "hole" in the graph.)

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \qquad x \neq 2$$

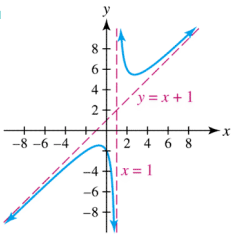


The graph is the line $y = x + 2$ with the point (2, 4) missing.

What is a Slant/Oblique Asymptote?

A third type of asymptote is neither ho or vertical.

Occurs when the numerator of a rational function has a degree *one more* than the degree of the denominator.



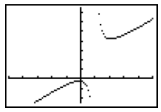
Example

Let $f(x) = \dfrac{x^2+1}{x-2}$.

a) Use a calculator to graph *f*.
b) Identify any asymptotes.
c) Sketch a graph of *f* that includes the asymptotes.

Solution

a)



Reminder: Slant/Oblique Asymptote occurs when the numerator of a rational function has a degree *one more* than the degree of the denominator.

Is that true in this rational function?

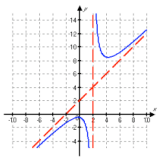
Example (Cont.)

Solution (Cont.)

b) Asymptotes: The function is undefined when $x - 2 = 0$ or when $x = 2$.

* Vertical asymptote at $x = 2$
* Oblique asymptote at $y = x + 2$

c)



How about horizontal asymptote? Why don't we have it in this rational function? How can you tell from the function itself?

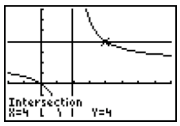
How to Solve Rational Equation?

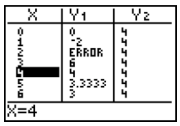
Solve $\dfrac{2x}{x-2} = 4.$

Solution

Symbolic *Graphical* *Numerical*

$$\frac{2x}{x-2} = 4$$
$$2x = 4(x-2)$$
$$2x = 4x - 8$$
$$-2x = -8$$
$$x = 4$$





Example

Solve $\dfrac{4}{x^2-1} = \dfrac{2}{x-1} + \dfrac{3}{x+1}.$

Solution

Multiply by the **LCD** to clear the fractions.

$$\frac{4}{x^2-1} = \frac{2}{x-1} + \frac{3}{x+1}$$
$$\frac{4(x-1)(x+1)}{x^2-1} = \frac{2(x-1)(x+1)}{x-1} + \frac{3(x-1)(x+1)}{x+1}$$
$$4 = 2(x+1) + 3(x-1)$$
$$4 = 2x + 2 + 3x - 3$$
$$4 = 5x - 1$$
$$1 = x$$

When 1 is substituted for x, two expressions in the given equation are undefined. There are no solutions.

What have we learned?

We have learned to:

1. Identify a rational function and state its domain.
2. Find and interpret vertical asymptotes.
3. Find and interpret horizontal asymptotes.
4. Solve rational equations.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition