MAC 1105

Module 1
Introduction to Functions and Graphs

Learning Objectives

Upon completing this module, you should be able to:

1. Recognize common sets of numbers.
2. Understand scientific notation and use it in applications.
3. Find the domain and range of a relation.
4. Graph a relation in the xy-plane.
5. Understand function notation.
6. Define a function formally.
7. Identify the domain and range of a function.
8. Identify functions.
9. Identify and use constant and linear functions.
10. Interpret slope as a rate of change.

Learning Objectives

11. Write the point-slope and slope-intercept forms for a line.

12. Find the intercepts of a line.

13. Write equations for horizontal, vertical, parallel, and perpendicular lines.

14. Write equations in standard form.

15. Identify and use nonlinear functions.

16. Recognize linear and nonlinear data.

17. Use and interpret average rate of change.

18. Calculate the difference quotient.

Introduction to Functions and Graphs

There are four major topics in this module:

- Functions and Models

- Graphs of Functions

- Linear Functions

- Equations of Lines

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Let's get started by recognizing some common set of numbers.

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What is the difference between Natural Numbers and Integers?

- Natural Numbers (or counting numbers)

   are numbers in the set $N = \{1, 2, 3, ...\}$.

- Integers are numbers in the set

   $I = \{... -3, -2, -1, 0, 1, 2, 3, ...\}$.

What are Rational Numbers?

Rational Numbers are real numbers which can be expressed as the ratio of two integers p/q where $q \neq 0$

Examples: $3 = 3/1$ $-5 = -10/2$ $0 = 0/2$

$0.5 = \frac{1}{2}$ $0.52 = 52/100$ $0.333\ldots = 1/3$

Note that:
- Every integer is a rational number.
- Rational numbers can be expressed as decimals
       which either terminate (end) or repeat a sequence
       of digits.

What are Irrational Numbers?

- Irrational Numbers are real numbers which are not rational numbers.

- Irrational numbers <u>Cannot</u> be expressed as the ratio of two integers.

- Have a decimal representation which <u>does not</u> terminate and <u>does not</u> repeat a sequence of digits.

Examples:

$$\sqrt{2}, \quad \sqrt[3]{5}, \quad \pi, \quad 0.01001000100001....$$

Classifying Real Numbers

Classify each number as one or more of the following: natural number, integer, rational number, irrational number.

$$\sqrt{25}, \quad \sqrt[3]{8}, \quad 3.14, \quad .01010101..., \quad \frac{22}{7}, \quad -\sqrt{11}$$

$\sqrt{25} = 5$ so it is a natural number, integer, rational number

$\sqrt[3]{8} = 2$ so it is a natural number, integer, rational number

3.14, $.01010101....$, and $\frac{22}{7}$ are rational numbers.

$-\sqrt{11}$ is an irrational number.

Let's Look at Scientific Notation

A real number *r* is in scientific notation

when *r* is written as *c* x 10^n, where

and *n* is an integer.

Examples:

The distance to the sun is 93,000,000 mi.

In scientific notation for this is 9.3×10^7 mi.

The size of a typical virus is .000005 cm.

In scientific notation for this is 5×10^{-6} cm.

What is a Relation?
What are Domain and Range?

A relation is a set of ordered pairs.

If we denote the ordered pairs by (x, y)

The set of all x – values is the DOMAIN.

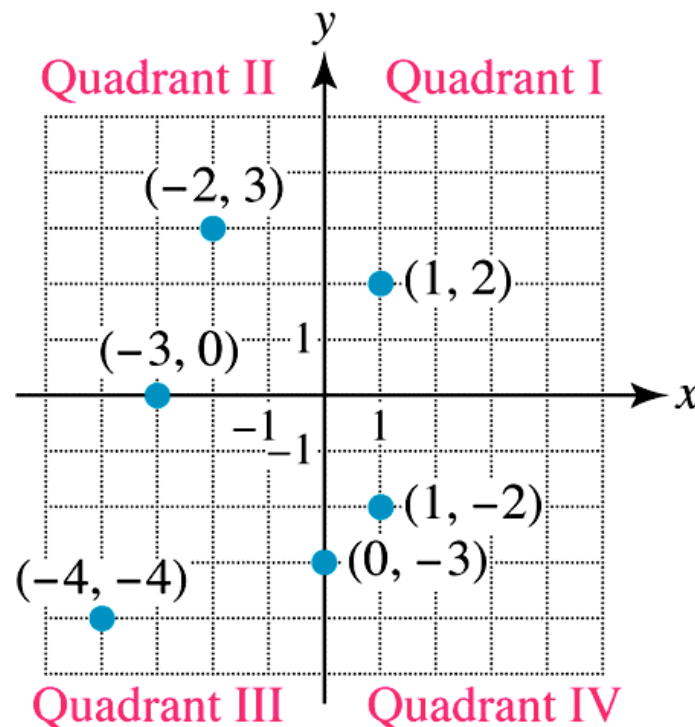The set of all y – values is the RANGE.

Example

The relation {(1, 2), (–2, 3), (–4, –4), (1, –2), (–3,0), (0, –3)}

has domain D = {–4, –3, –2, 0, 1}

and range R = {–4, –3, –2, 0, 2, 3}

How to Represent
a Relation in a Graph?

The relation {(1, 2), (–2, 3), (–4, –4), (1, –2), (–3, 0), (0, –3)}
has the following graph:

Is Function a Relation?

Recall that a relation is a set of ordered pairs (x,y) .

If we think of values of *x* as being inputs and values of *y* as being outputs, a function is a relation such that for each input there is <u>exactly one</u> output.

This is symbolized by *output* = *f*(input) or

$$y = f(x)$$

Function Notation

$y = f(x)$

- Is pronounced "y is a function of x."

- Means that given a value of x (input), there is exactly one corresponding value of y (output).

- x is called the independent variable as it represents inputs, and y is called the dependent variable as it represents outputs.

- Note that: $f(x)$ is NOT f multiplied by x. f is NOT a variable, but the name of a function (the name of a relationship between variables).

What are Domain and Range?

The set of all meaningful inputs is called the DOMAIN of the function.

The set of corresponding outputs is called the RANGE of the function.

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What is a Function?

A **function** is a relation in which each element of the domain corresponds to exactly one element in the range.

The function may be defined by a set of ordered pairs, a table, a graph, or an equation.

Here is an Example

- Suppose a car travels at 70 miles per hour. Let y be the distance the car travels in x hours. Then $y = 70\,x$.

- Since for each value of x (that is the time in hours the car travels) there is just one corresponding value of y (that is the distance traveled), y is a function of x and we write

$$y = f(x) = 70x$$

- Evaluate f(3) and interpret.
  - f(3) = 70(3) = 210. This means that the car travels 210 miles in 3 hours.

Here is Another Example

- Given the following data, is *y* a function of *x*?

  - Input   *x*   3   4   8
  - Output *y*   6   6   − 5

- Note: The data in the table can be written as the set of ordered pairs {(3,6), (4,6), (8, −5)}.

- *Yes, y* is a function of *x*, because for each value of *x*, there is just one corresponding value of *y*. Using function notation we write *f*(3) = 6; *f*(4) = 6; *f*(8) = −5.

One More Example

- Undergraduate Classification at Study-Hard University (SHU) is a function of Hours Earned. We can write this in function notation as $C = f(H)$.

- Why is C a function of H?

  – For each value of H there is <u>exactly one</u> corresponding value of C.

  – In other words, for each input there is <u>exactly one</u> corresponding output.

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One More Example (Cont.)

- Here is the classification of students at SHU (from catalogue):

- No student may be classified as a sophomore until after earning at least 30 semester hours.

- No student may be classified as a junior until after earning at least 60 hours.

- No student may be classified as a senior until after earning at least 90 hours.

One More Example (Cont.)

- Remember $C = f(H)$

- Evaluate $f(20)$, $f(30)$, $f(0)$, $f(20)$ and $f(61)$:

    – $f(20)$ = **Freshman**
    – $f(30)$ = **Sophomore**
    – $f(0)$ = **Freshman**
    – $f(61)$ = **Junior**

- What is the domain of f?
- What is the **range** of f?

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One More Example (Cont.)

Domain of f is the set of <u>non-negative integers</u> $\{0,1,2,3,4...\}$

> Alternatively, some individuals say the domain is the set of <u>positive rational numbers</u>, since technically one could earn a fractional number of hours if they transferred in some quarter hours. For example, 4 quarter hours = 2 2/3 semester hours.

> Some might say the domain is the set of <u>non-negative</u> <u>real numbers</u>$[0, \infty)$, but this set includes irrational numbers. It is impossible to earn an irrational number of credit hours. For example, one could not earn $\sqrt{2}$ hours.

Range of *f* is {Fr, Soph, Jr, Sr}

Identifying Functions

Referring to the previous example concerning SHU, is hours earned a function of classification? That is, is $H = f(C)$? Explain why or why not.

Is classification a function of years spent at SHU? Why or why not?

- Given $x = y^2$, is y a function of x? Why or why not?
- Given $x = y^2$, is x a function of y? Why or why not?
- Given $y = x^2 + 7$, is y a function of x? Why, why not?

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Identifying Functions (Cont.)

- Is hours earned a function of classification? That is, is $H = f(C)$?

- That is, for each value of C is there just one corresponding value of H?

    - No. One example is

        - if C = Freshman, then H could be 3 or 10 (or lots of other values for that matter)

Identifying Functions (Cont.)

- Is classification a function of years spent at SHU? That is, is $C = f(Y)$?

- That is, for each value of Y is there just one corresponding value of C?

  - No. One example is

    - if $Y = 4$, then C could be Sr. or Jr. It could be Jr if a student was a part time student and full loads were not taken.

Identifying Functions (Cont.)

- Given $x = y^2$, is y a function of x?

- That is, given a value of x, is there just one corresponding value of y?

    – No, if $x = 4$, then $y = 2$ or $y = -2$.

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Identifying Functions (Cont.)

- Given $x = y^2$, is x a function of y?

- That is, given a value of y, is there just one corresponding value of x?

  - Yes, given a value of y, there is just one corresponding value of x, namely y^2.

Identifying Functions (Cont.)

- Given $y = x^2 + 7$, is y a function of x?

- That is, given a value of x, is there just one corresponding value of y?

  - Yes, given a value of x, there is just one corresponding value of y, namely $x^2 + 7$.

Five Ways to Represent
a Function

- Verbally

- Numerically

- Diagrammaticly

- Symbolically

- Graphically

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Verbal Representation

- Referring to the previous example:

  – If you have less than 30 hours, you are a freshman.

  – If you have 30 or more hours, but less than 60 hours, you are a sophomore.

  – If you have 60 or more hours, but less than 90 hours, you are a junior.

  – If you have 90 or more hours, you are a senior.

Numeric Representation

H	C
0	Freshman
1	Freshman
?	
?	
?	
?	
29	Freshman
30	Sophomore
31	Sophomore
?	
?	
?	
59	Sophomore
60	Junior
61	Junior
?	
?	
?	
89	Junior
90	Senior
91	Senior
?	
?	
?	

Symbolic Representation

$$C = f(H) = \begin{cases} \text{Freshman} & \text{if } 0 \leq H < 30 \\ \text{Sopho} & \text{if } 30 \leq H < 60 \\ \text{Junior} & \text{if } 60 \leq H < 90 \\ \text{Senior} & \text{if } H \geq 90 \end{cases}$$

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Diagrammatic Representation

H

C

0
1
2
•
•
•
29
30
31
•
•
•
59
60
61
•
•
•
89
90
91
•
•
•

Freshman

Sophomore

Junior

Senior

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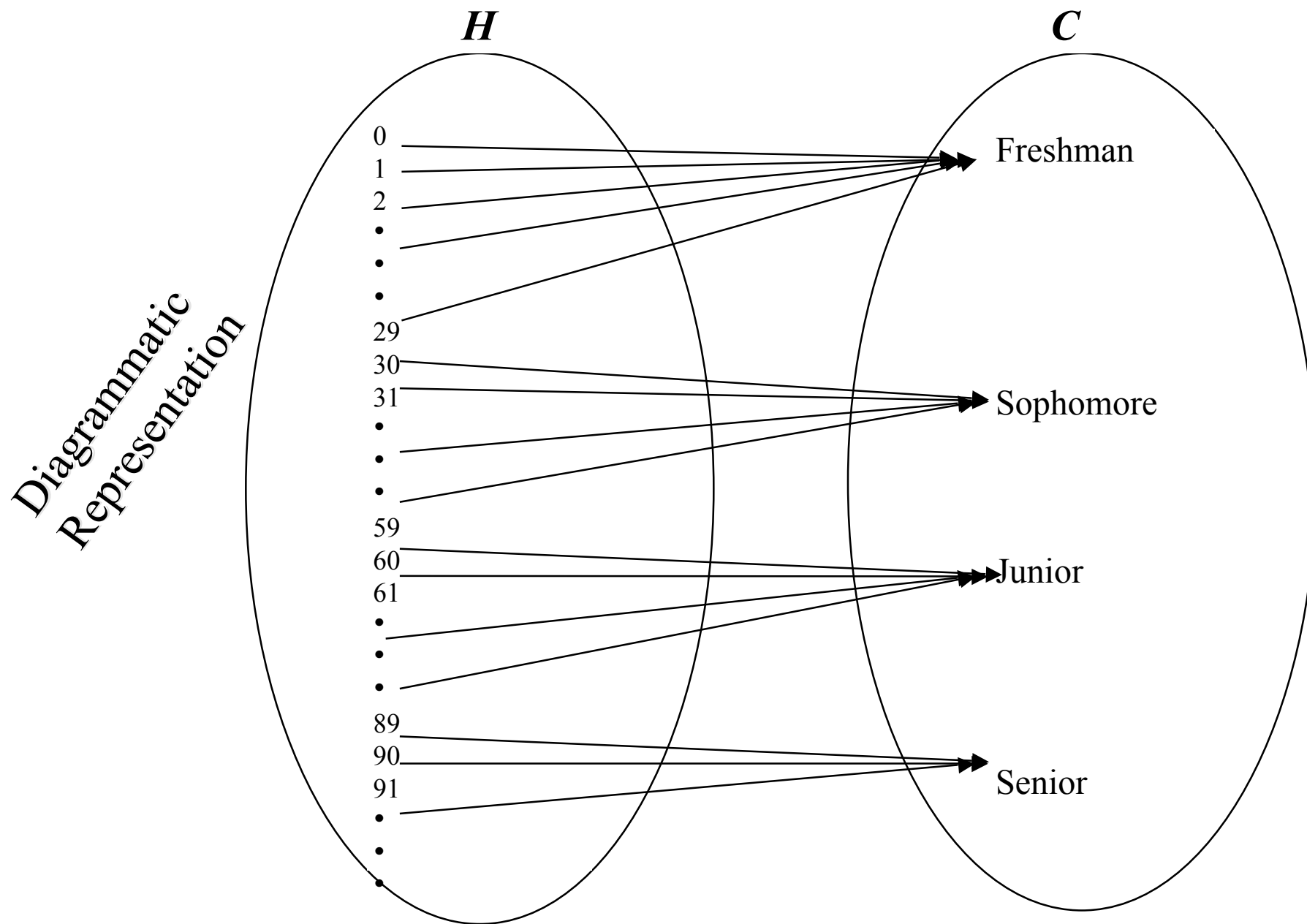
Graphical Representation

- In this graph the domain is considered to be $[0, \infty)$

- instead of {0,1,2,3…}, and note that inputs are typically graphed on the horizontal axis and outputs are typically graphed on the vertical axis.
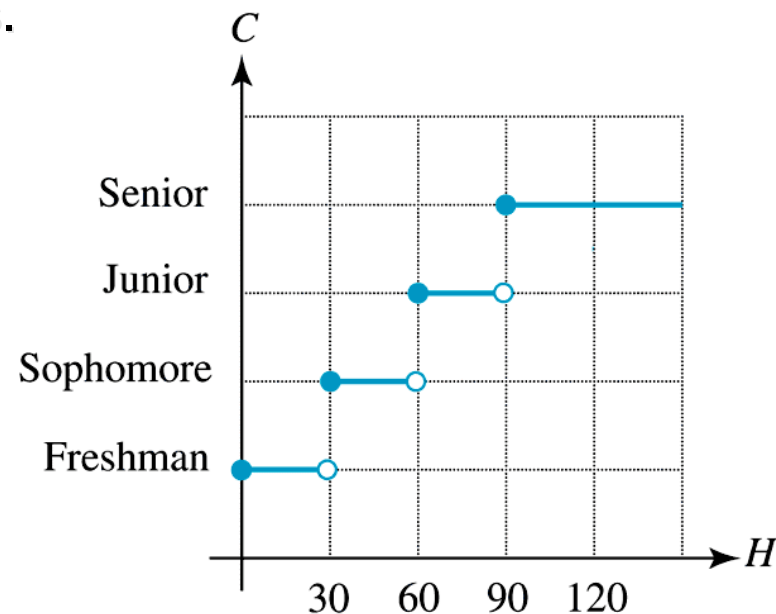
Vertical Line Test

- Another way to determine if a graph represents a function, simply visualize vertical lines in the *xy*-plane. If each vertical line intersects a graph at no more than one point, then it is the graph of a function.

What is a Constant Function?

A function f represented by $f(x) = b$,

where b is a constant (fixed number), is a

constant function.

Examples:

$$f(x) = 2$$

$$f(x) = \frac{-1}{2}$$

$$f(x) = \sqrt{2}$$

$f(x) = 2$
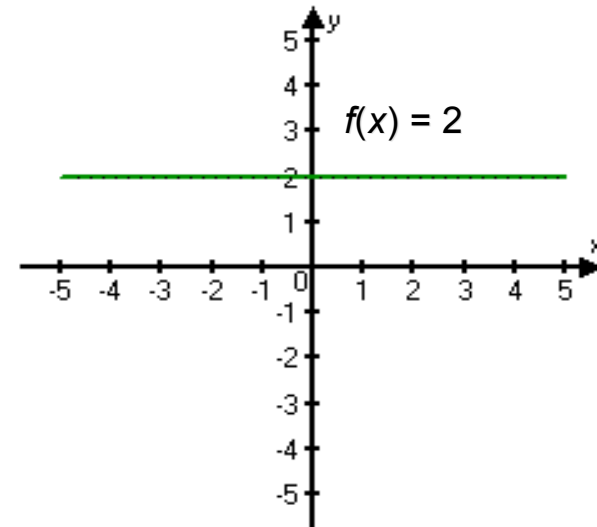
Note: Graph of a constant function is a horizontal line.

What is a Linear Function?

A function f represented by $f(x) = ax + b$,

  where a and b are constants, is a linear function.

(It will be an identity function, if constant $a = 1$ and constant $b = 0$.)

Examples:

$f(x) = 2x + 3$ (Note: $a = 2$ and $b = 3$)

$f(x) = -5x - \dfrac{1}{2}$ $\left(\text{Note: } a = -5 \text{ and } b = -\dfrac{1}{2}\right)$

$f(x) = 2$ (Note: $a = 0$ and $b = 2$)
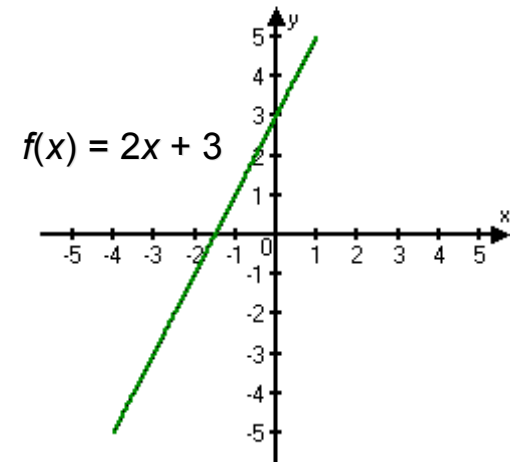
$f(x) = 2x + 3$

Note that a $f(x) = 2$ is both a linear function and a constant function.
A constant function is a special case of a linear function.

Rate of Change
of a Linear Function

x	*y*
−2	−1
−1	1
0	3
1	5
2	7
3	9

Table of values for $f(x) = 2x + 3$.

- Note throughout the table, as x increases by 1 unit, y increases by 2 units. In other words, the RATE OF CHANGE of y with respect to x is constantly 2 throughout the table. Since the rate of change of y with respect to x is constant, the function is LINEAR. Another name for rate of change of a linear function is SLOPE.

The Slope of a Line

- The slope m of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(x_2, y_2)

(x_1, y_1)

$\Delta y = y_2 - y_1$

$\Delta x = x_2 - x_1$

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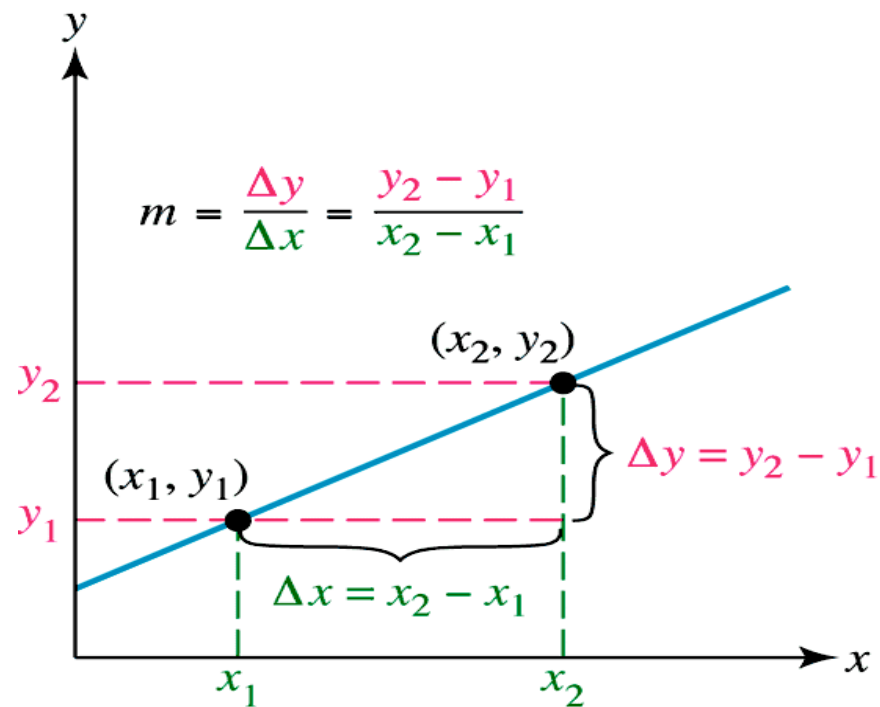
Example of Calculation of Slope

- Find the slope of the line passing through the points (–2, –1) and (3, 9).

$$m = \frac{\Delta y}{\Delta x} = \frac{9-(-1)}{3-(-2)} = \frac{10}{5} = 2$$
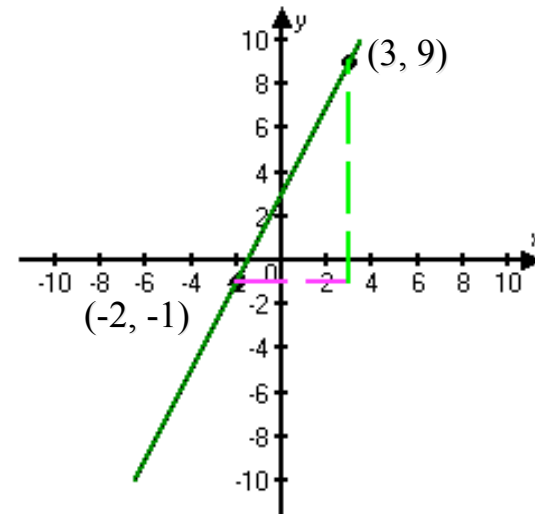
- The slope being 2 means that for each unit x increases, the corresponding increase in *y* is 2. The rate of change of *y* with respect to x is 2/1 or 2.

How to Write the Point-Slope Form of the Equation of a Line?

The line with slope m passing through the point (x_1, y_1) has equation

$$y = m(x - x_1) + y_1$$

or

$$y - y_1 = m(x - x_1)$$

How to Write the Equation of the Line Passing Through the Points (–4, 2) and (3, –5)?

To write the equation of the line using **point-slope form**

$$y = m\,(x - x_1) + y_1$$

the **slope** m and a **point** (x_1, y_1) are needed.

Let $(x_1, y_1) = (3, -5)$.

Calculate m using the two given points.

$$m = \frac{\ddot{A}y}{\ddot{A}x} = \frac{-5 - 2}{3 - (-4)} = \frac{-7}{7} = -1$$

Equation is $\qquad y = -1\,(x - 3) + (-5)$

This simplifies to $\qquad y = -x + 3 + (-5)$

$$y = -x - 2$$

Slope-Intercept Form

The line with slope m and y-intercept b is given by

- $y = m\,x + b$

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How to Write the Equation of a line passing through the point (0,-2) with slope ½?

Since the point $(0, -2)$ has an *x*-coordinate of 0, the point is a *y*-intercept. Thus $b = -2$

Using **slope-intercept form**

$$y = m\,x + b$$

the equation is

$$y = (½)\,x - 2$$
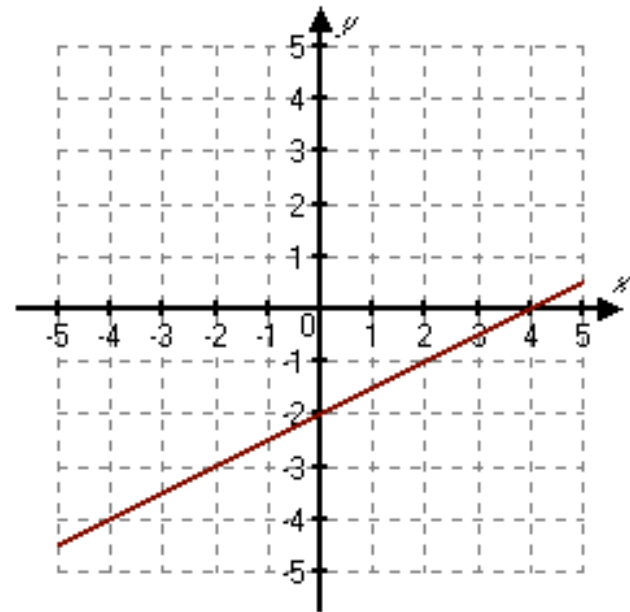
How to Write an Equation of a Linear Function in Slope-Intercept Form?

- What is the slope?
  - As x increases by 4 units, y decreases by 3 units so the slope is $-3/4$

- What is the y-intercept?
  - The graph crosses the y–axis at (0,3) so the y–intercept is 3.

- What is the equation?
  - Equation is
  - $f(x) = (-\frac{3}{4})x + 3$
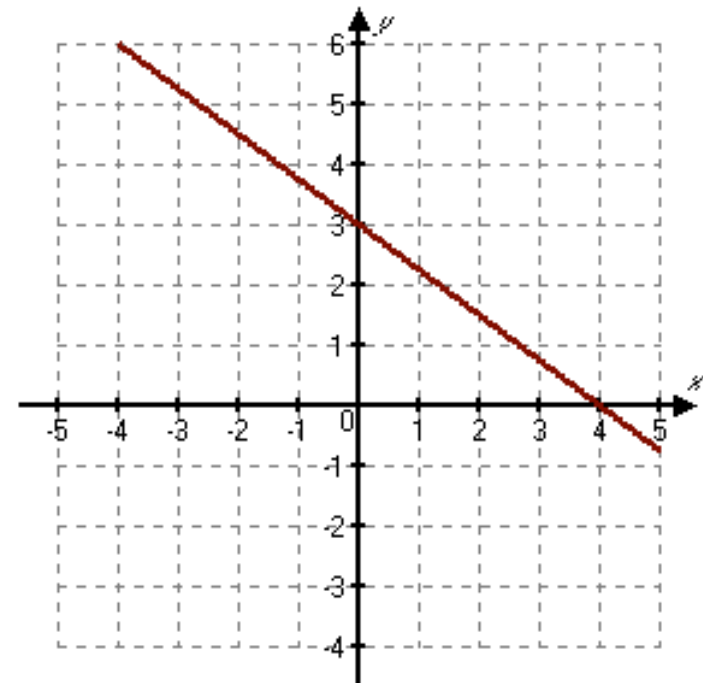
What is the Standard Form for the Equation of a Line?

$$ax + by = c$$

is standard form (or general form) for the equation of a line.

How to Find *x*-Intercept and *y*-intercept?

- To find the *x*-intercept, let *y* = 0 and solve for *x*.
  - $2x - 3(0) = 6$
  - $2x = 6$
  - $x = 3$
- To find the *y*-intercept, let *x* = 0 and solve for *y*.
  - $2(0) - 3y = 6$
  - $-3y = 6$
  - $y = -2$

(0, 2)

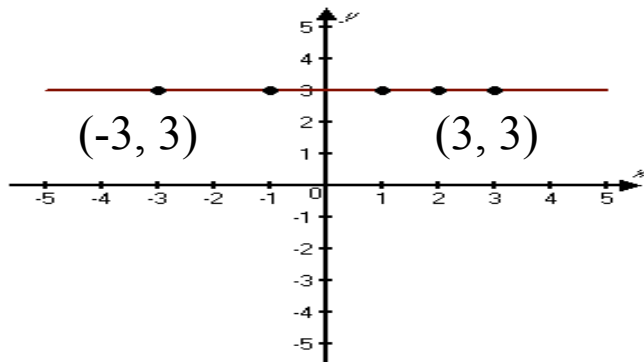(3, 0)
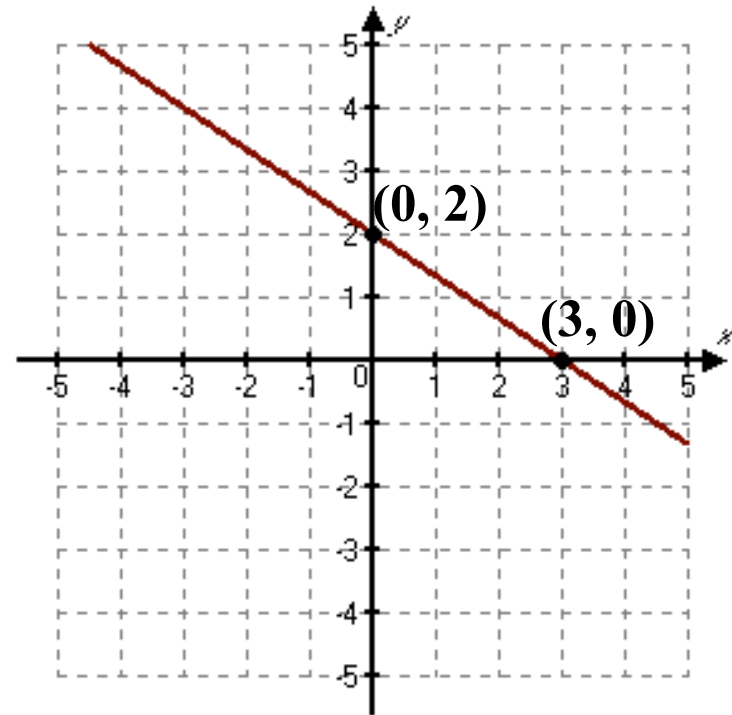
What are the Characteristics of Horizontal Lines?

Slope is 0, since $\Delta y = 0$ and $m = \Delta y / \Delta x$

Equation is: $y = mx + b$
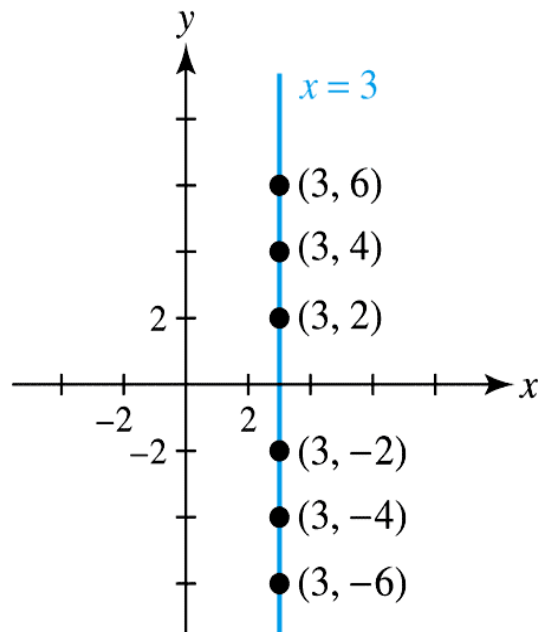
$$y = (0)x + b$$

$$y = b \text{ where } b \text{ is the } y\text{-intercept}$$

Example: $y = 3$ (or $0x + y = 3$)

(-3, 3)   (3, 3)

Note that regardless of the value of x, the value of y is always 3.

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# What are the Characteristics of Vertical Lines?

- Slope is undefined, since $\Delta x = 0$ and $m = \Delta y / \Delta x$

- Example:

- Note that regardless of the value of y, the value of x is always 3.
- Equation is $x = 3$ (or $x + 0y = 3$)
- Equation of a vertical line is $x = k$ where k is the x-intercept.

What Are the Differences Between Parallel and Perpendicular Lines?

- Parallel lines have the same slant, thus they have the same slopes.
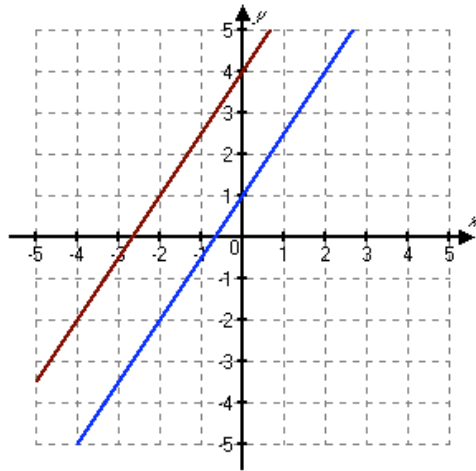
Perpendicular lines have slopes which are negative reciprocals (unless one line is vertical!)

$$y = \frac{3}{2}x + 4$$

$$y = -\frac{2}{3}x + 2$$

$(-2, 1)$
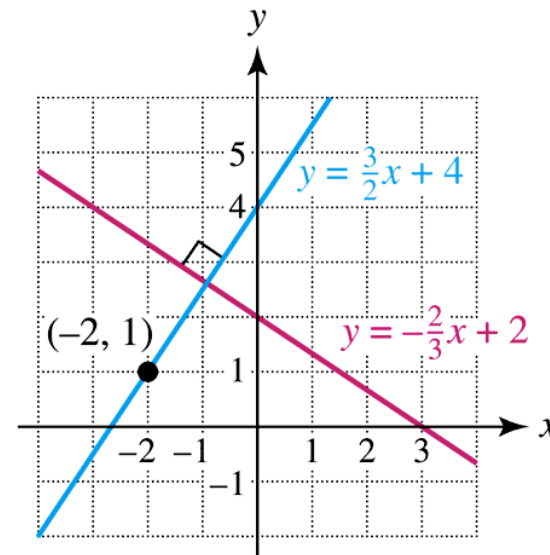
How to Find the Equation of the Line Perpendicular to y = -4x - 2 Through the Point (3,-1)?

The slope of any line perpendicular to $y = -4x - 2$ is ¼

(-4 and ¼ are negative reciprocals)

Since we know the slope of the line and a point on the line we can use point-slope form of the equation of a line:

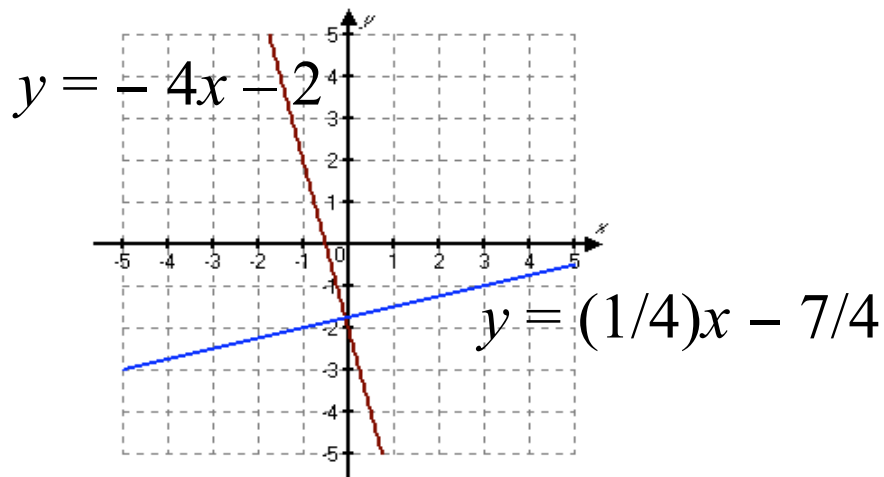$y = m(x - x_1) + y_1$

$y = (1/4)(x - 3) + (-1)$

$$y = -4x - 2$$

In slope-intercept form:

$y = (1/4)x - (3/4) + (-1)$

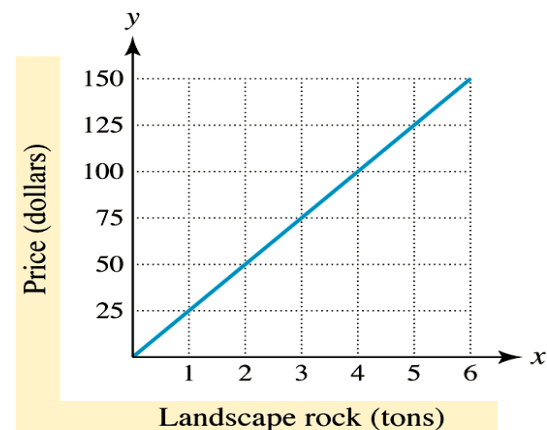$y = (1/4)x - 7/4$

$$y = (1/4)x - 7/4$$

Example of a Linear Function

The table and corresponding graph show the price *y* of *x* tons of landscape rock.

X (tons)	*y* (price in dollars)
	25
	5
	75
4	100

y is a linear function of *x* and the slope is $\dfrac{\Delta y}{\Delta x} = \dfrac{50-25}{2-1} = 25$

The rate of change of price *y* with respect to tonage *x* is 25 to 1.

This means that for an increase of 1 ton of rock the price increases by $25.

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Example of a Nonlinear Function

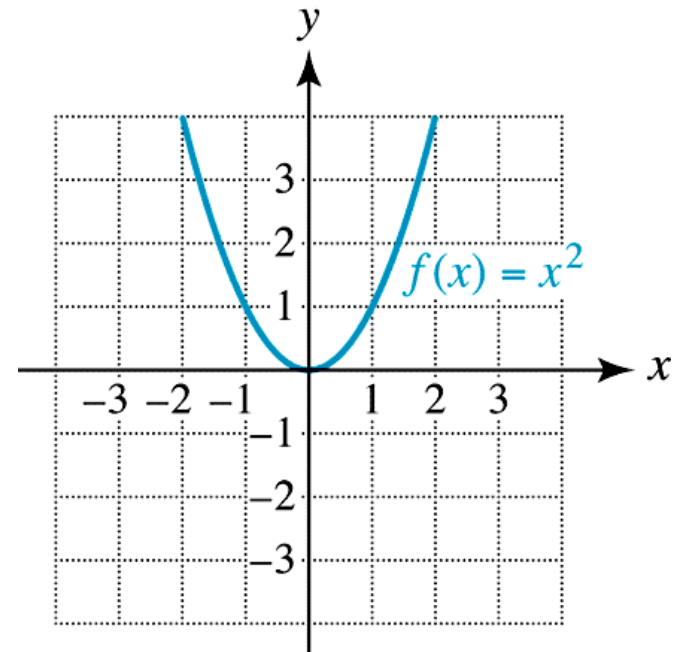| x	y
0	0
1	1
2	4

$f(x) = x^2$

Table of values for $f(x) = x^2$

Note that as x increases from 0 to 1, y increases by 1 unit; while as x increases from 1 to 2, y increases by 3 units. 1 does not equal 3. This function does NOT have a CONSTANT RATE OF CHANGE of y with respect to x, so the function is NOT LINEAR.

Note that the graph is not a line.

Average Rate of Change

Let (x_1, y_1) and (x_2, y_2) be distinct points on the graph of a function f. The average rate of change of f from x_1 to x_2 is
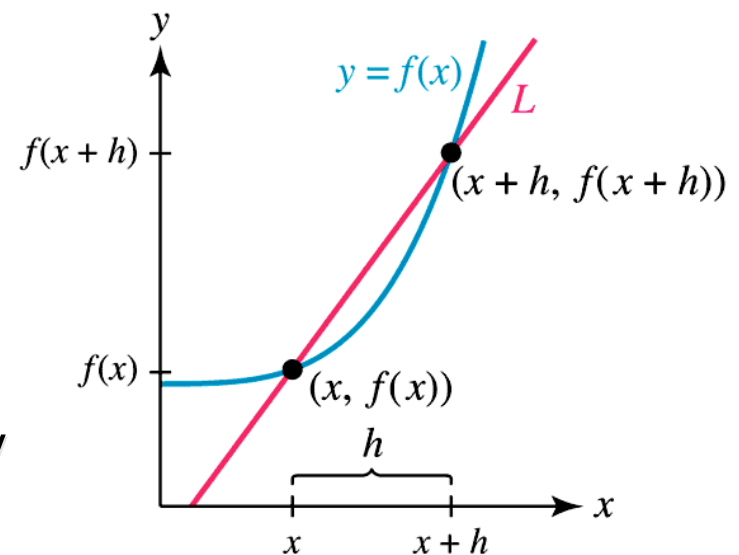
$$\frac{y_2 - y_1}{x_2 - x_1}$$

Note that the average rate of change of f from x_1 to x_2 is the slope of the line passing through (x_1, y_1) and (x_2, y_2)

What is the Difference Quotient?

The difference quotient of a function f is an expression of the form
$$\frac{f(x+h) - f(x)}{h}$$
where h is not 0.

Note that a difference quotient is actually an average rate of change.

What have we learned?

We have learned to:

1. Recognize common sets of numbers.
2. Understand scientific notation and use it in applications.
3. Find the domain and range of a relation.
4. Graph a relation in the xy-plane.
5. Understand function notation.
6. Define a function formally.
7. Identify the domain and range of a function.
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9. Identify and use constant and linear functions.
10. Interpret slope as a rate of change.

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What have we learned? (Cont.)

11. Write the point-slope and slope-intercept forms for a line.

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13. Write equations for horizontal, vertical, parallel, and perpendicular lines.

14. Write equations in standard form.

15. Identify and use nonlinear functions.

16. Recognize linear and nonlinear data.

17. Use and interpret average rate of change.

18. Calculate the difference quotient.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition

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