

MAC 1105

Module 2

Modeling Linear Functions

Learning Objectives

Upon completing this module, you should be able to:

1. Recognize linear equations.
2. Solve linear equations symbolically and graphically.
3. Find the zeros of a function.
4. Identify solutions, zeros, and x-intercept.
5. Solve an equation for a specified variable.

Learning Objectives

6. Identify a table of values for a linear function.
7. Use constant first differences.
8. Model data with a linear function.
9. Use linear regression to model data.
10. Apply problem-solving strategies.

Modeling Linear Functions

There are two major topics in this module:

- Algebraic and Graphical Solutions of Linear Equations
- Fitting Lines to Data Points: Modeling Linear Functions

Linear Equations in One Variable

A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$ where a and b are real numbers with $a \neq 0$. (Note the power of x is always 1.)

Examples of **linear equations** in one variable:

$$5x + 4 = 2 + 3x \text{ simplifies to } 2x + 2 = 0$$

Note the power of x is always 1.

$$-1(x - 3) + 4(2x + 1) = 5 \text{ simplifies to } 7x + 2 = 0$$

Note the power of x is always 1.

Examples of equations in one variable which are not linear:

$$x^2 = 1 \text{ (Note the power of } x \text{ is NOT 1.)}$$

$$\frac{1}{x-1} + x = 0 \text{ (Note the power of } x \text{ is NOT always 1.)}$$

How to Solve a Linear Equations Symbolically?

Solve $-1(x - 3) + 4(2x + 1) = 5$ for x

$$-1x + 3 + 8x + 4 = 5$$

$$7x + 7 = 5$$

$$7x = 5 - 7$$

$$7x = -2$$

$$x = -2/7 \quad \text{“Exact Solution”}$$

Linear Equations can always be solved symbolically and will produce an EXACT SOLUTION.

The solution procedure is to isolate the variable on the left in a series of steps in which the same quantity is added to or subtracted from each side and/or each side is multiplied or divided by the same non-zero quantity. This is true because of the addition and multiplication properties of equality.

How to Solve a Linear Equation Involving Fractions Symbolically?

Solve $\frac{x-1}{3} + 5 = \frac{1}{4}$
Solution Process:

$$12\left(\frac{x-1}{3} + 5\right) = 12\left(\frac{1}{4}\right)$$

$$4(x-1) + 60 = 3$$

$$4x - 4 + 60 = 3$$

$$4x + 56 = 3$$

$$4x = 3 - 56$$

$$4x = -53$$

$$x = \frac{-53}{4} = -13.25$$

- When solving a linear equation involving fractions, it is often helpful to **multiply both sides by the least common denominator** of all of the denominators in the equation.
- The least common denominator of 3 and 4 is 12.

Note that this is another “Exact Solution.”

How to Solve a Linear Equation Graphically?

Solve $\frac{x-1}{3} + 5 = \frac{1}{4}$

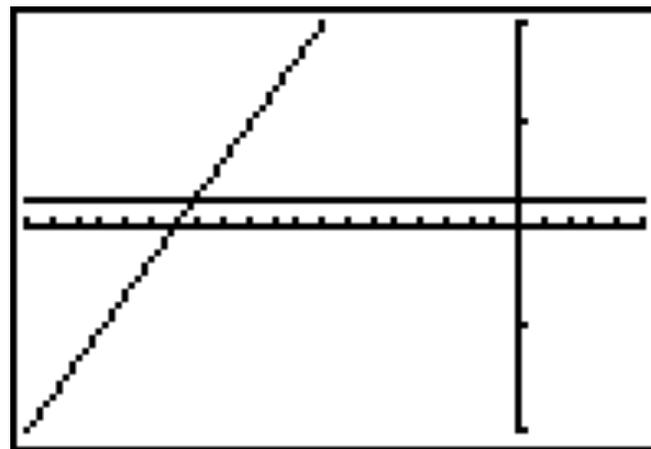
$[-20, 5, 1]$ by $[-2, 2, 1]$

Solution Process:

Graph $y_1 = \frac{x-1}{3} + 5$

$$y_2 = \frac{1}{4}$$

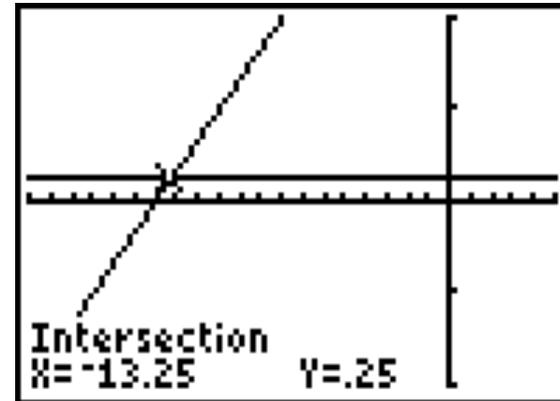
in a window in which
the graphs intersect.



How to Solve a Linear Equation Graphically?(Cont.)

- Locate **points of intersection**. x-coordinates of points of intersection are **solutions** to the equation.

$[-20, 5 \ 1]$ by $[-2, 2, 1]$

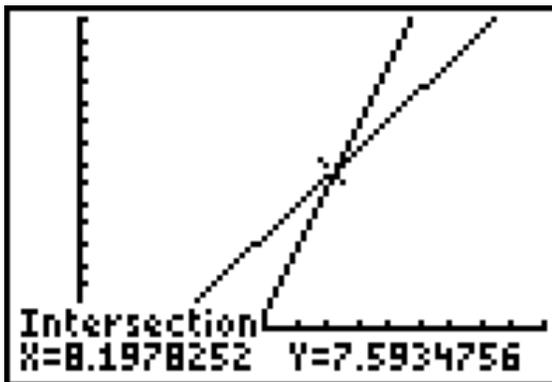
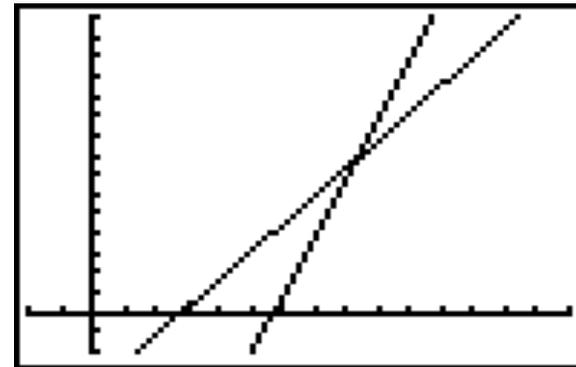
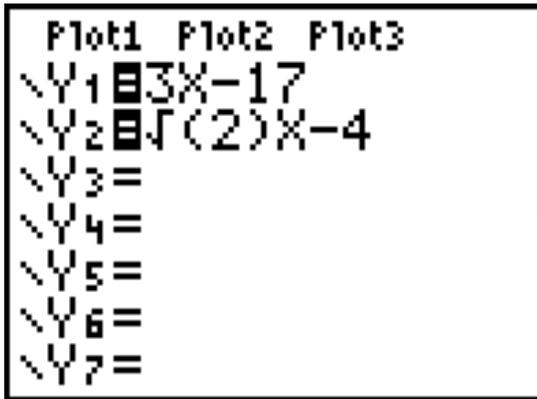


The solution to the equation is -13.25 .

This agrees exactly with the solution produced from the symbolic method. Sometimes a graphical method will produce only an **approximate solution**.

Another Example

Solve $3x - 17 = \sqrt{2}x - 4$ $[-2, 15, 1]$ by $[-2, 15, 1]$



Approximate solution (to the nearest hundredth) is 8.20. The exact solution of $\frac{13}{3-\sqrt{2}}$ can be found by solving symbolically.

What Are the Four Steps in Modeling with Linear Equations?

STEP 1: Read the problem and make sure you understand it. Assign a variable to what you are being asked. If necessary, write other quantities in terms of the variable.

STEP 2: Write an equation that relates the quantities described in the problem. You may need to sketch a diagram and refer to known formulas.

STEP 3: Solve the equation and determine the solution.

STEP 4: Look back and check your solution. Does it seem reasonable?

Example of Modeling with Linear Equations

In 2 hours an athlete travels 18.5 miles by running at 11 miles per hour and then by running at 9 miles per hour. How long did the athlete run at each speed?

STEP 1: We are asked to find the time spent running at each speed. If we let x represent the time in hours running at 11 miles per hour, then $2 - x$ represents the time spent running at 9 miles per hour.

x : Time spent running at 11 miles per hour

$2 - x$: Time spent running at 9 miles per hour

STEP 2: Distance d equals rate r times time t : that is, $d = rt$. In this example we have two rates (speeds) and two times. The total distance must sum to 18.5 miles.

$$d = r_1t_1 + r_2t_2$$

$$18.5 = 11x + 9(2 - x)$$

Example of Modeling with Linear Equations (Cont.)

STEP 3: Solving $18.5 = 11x + 9(2 - x)$ symbolically

$$18.5 = 11x + 18 - 9x$$

$$18.5 - 18 = 2x$$

$$.5 = 2x$$

$$x = .5/2$$

$$x = .25$$

The athlete runs .25 hours (15 minutes) at 11 miles per hour and 1.75 hours (1 hours and 45 minutes) at 9 miles per hour.

STEP 4: We can check the solution as follows.

$$11(.25) + 9(1.75) = 18.5 \quad (\text{It checks.})$$

This sounds reasonable. The average speed was 9.25 mi/hr, that is 18.5 miles/2 hours. Thus the runner would have to run longer at 9 miles per hour than at 11 miles per hour, since 9.25 is closer to 9 than 11.

One More Linear Equations Problem

Pure water is being added to a 25% solution of 120 milliliters of hydrochloric acid. How much water should be added to reduce it to a 15% mixture?

STEP 1: We need the amount of water to be added to 120 milliliters of 25% acid to make a 15% solution. Let this amount of water be equal to x .

x : Amount of pure water to be added

$x + 120$: Final volume of 15% solution

STEP 2: The total amount of acid in the solution after adding the water must equal the amount of acid before the water is added. The volume of pure acid after the water is added equals 15% of $x + 120$ milliliters, and the volume of pure acid before the water is added equals 25% of 120 milliliters. So we must solve the equation

$$.15(x + 120) = .25(120)$$

One More Linear Equations Problem (Cont.)

STEP 3: Solving $.15(x + 120) = .25(120)$ symbolically

$$.15x + 18 = 30$$

$$.15x = 12$$

$$x = 12/.15$$

$$x = 80 \text{ milliliters}$$

STEP 4: This sounds reasonable. If we added 120 milliliters of water, we would have diluted the acid to half its concentration, which would be 12.5%. It follows that we should not add much as 120 milliliters since we want a 15% solution.

Steps for Solving a Linear Equation in One Variable

- Step 1: **Eliminate the fractions** (if any): Multiply both sides by the least common denominator.
- Step 2: **Clear parentheses** (if any) and combine like terms.
- Step 3: **Get all terms with the variable on one side** of the equation and all numbers on the other side.
- Step 4: Get an equation with **just the variable on one side of the equation**. Divide both sides of the equation by the coefficient of the variable, if necessary.
- Step 5: **Check the solution** by substitution in the original equation. Does it seem reasonable?

The Zeros of a Function

Any number a , for which $f(a) = 0$, is called a zero of the function $f(x)$.

a is the zero of the function f .

If a is real, a is the x -intercept of the graph of the function f .

a is the solution to the equation $f(x) = 0$.

Example:

The zero of $f(x) = 5x - 20$ is 4 , because $f(4)=0$.

4 is the zero of the function f .

Since 4 is real, 4 is also the x -intercept of the graph of the function f .

4 is the solution to the equation $f(x) = 0$.

Example of An Exact Model

The function $f(x) = 2.1x - 7$ models the data in table **exactly**.

x	-1	0	1	2
y	-9.1	-7	-4.9	-2.8

Note that:

- $f(-1) = 2.1(-1) - 7 = -9.1$ (Agrees with value in table)
- $f(0) = 2.1(0) - 7 = -7$ (Agrees with value in table)
- $f(1) = 2.1(1) - 7 = -4.9$ (Agrees with value in table)
- $f(2) = 2.1(2) - 7 = -2.8$ (Agrees with value in table)

Note the differences between each consecutive outputs are always -2.1 for equally spaced inputs. There is a name for these constant differences. It's called the Constant First Differences.

Example of An Approximate Model

The function $f(x) = 5x + 2.1$ models the data in table **approximately**.

x	-1	0	1
y	-2.9	2.1	7

Note that:

- $f(-1) = 5(-1) + 2.1 = -2.9$ (Agrees with value in table)
- $f(0) = 5(0) + 2.1 = 2.1$ (Agrees with value in table)
- $f(1) = 5(1) + 2.1 = 7.1$ 7 (Value is **approximately** the value in the table, but not **exactly**.)

Note the differences between each consecutive outputs for equally spaced inputs are “nearly constant.”

Constant First Differences and Linear Model

- If the **first differences** of data outputs are **constant** (for equally spaced or uniform inputs), a linear model can be found that **fits** the data **exactly**.
- If the **first differences** of data outputs are **nearly constant** (for equally spaced or uniform inputs), a linear model can only be found by an **approximate fit** for the data.

Discrete and Continuous

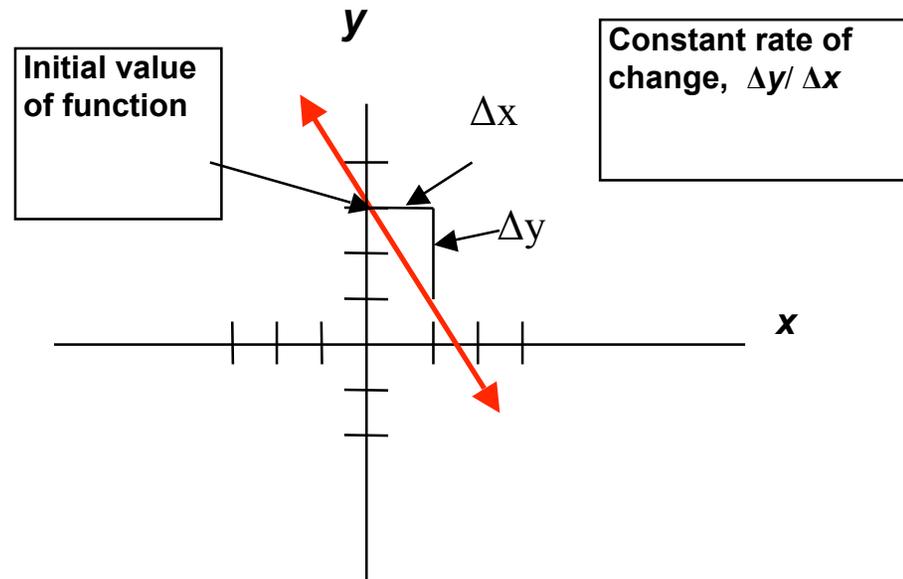
- **Discrete:** It is used to describe the data or a function that is presented in the form of a table or in a scatterplot.
- **Continuous:** It is used to describe a function or graph when the **inputs can be any real number.**

Modeling with Linear Functions

A linear function, $f(x) = mx + b$, has a constant rate of change, that is a constant slope.

$$f(0) = m(0) + b = b.$$

When the input of the function is 0, the output is b . So the y – intercept b is sometimes called the initial value of the function.



Note that a linear function is always a straight line.

The Linear Function Model

To model a quantity that is changing at a **constant rate**, the following may be used.

$$f(x) = (\text{constant rate of change})x + \text{initial amount}$$

Because

constant rate of change corresponds to the **slope**, and
initial amount corresponds to the **y – intercept**;

this is simply $f(x) = mx + b$

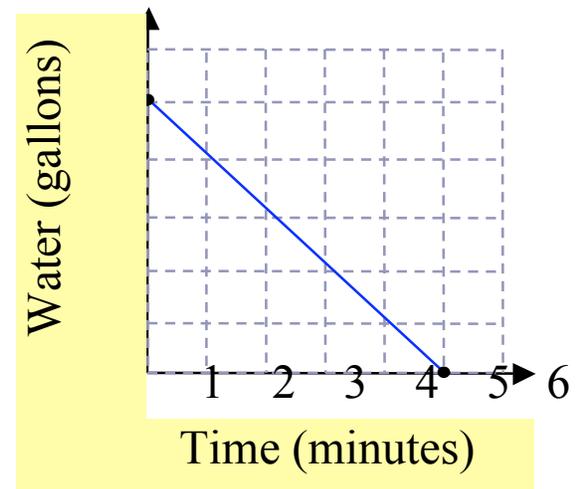
Example of Modeling with a Linear Function

A 50-gallon tank is initially full of water and being drained at a constant rate of 10 gallons per minute. Write a formula that models the number of gallons of water in the tank after x minutes.

The water in the tank is changing at a constant rate, so the linear function model $f(x) = (\text{constant rate of change})x + \text{initial amount}$ applies.

So $f(x) = (-10 \text{ gal/min})(x \text{ min}) + 50 \text{ gal}$.

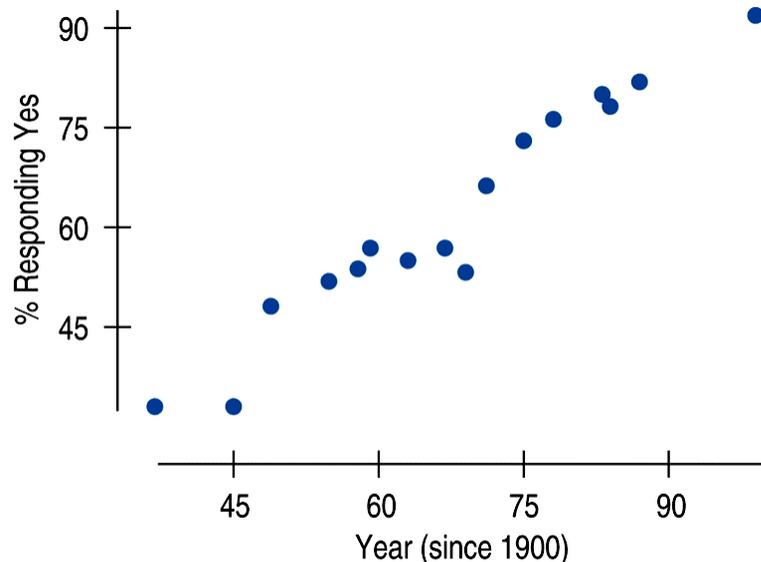
Without specifically writing the units, this is $f(x) = -10x + 50$



What is a Scatterplot?

Scatterplot may be the most common and most effective display for data.

- In a scatterplot, you can see patterns, trends, relationships, and even the occasional extraordinary value sitting apart from the others.



- This figure shows a **positive association** between the year since 1900 and the % of people who say they would vote for a woman president.
- As the years have passed, the percentage who would vote for a woman has increased.

The Linear Model

The **linear model** is just **an equation of a straight line** through the data.

- The points in the scatterplot don't all line up, but a straight line can summarize the general pattern.
- The linear model can help us understand how the values are **associated**.

The Linear Model and the Predicted Value

- The **model won't be perfect**, regardless of the line we draw.
- Some points will be above the line and some will be below.
- The **estimate made** from a model is the **predicted value**.

The Least-Squares Regression Line

- We approximate our **linear model** with
$$f(x) = ax + b$$
- This model says that our *predictions* from our model follow a **straight line**.
- If the model is a good one, the data values will scatter closely around it.

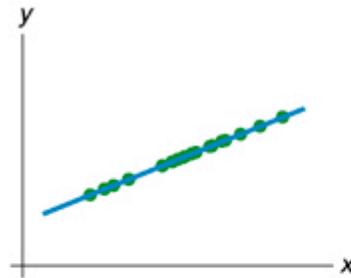
What is Linear Regression?

- Linear regression is a procedure which defines the best-fit line as the line which the sum of the squares of vertical distances from the data points to the line is a minimum.
- The linear regression procedure is also called the least-squares method.

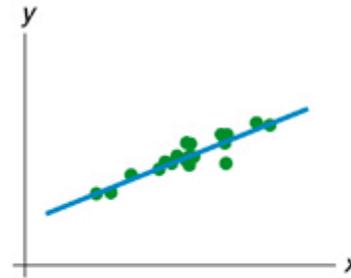
What is Correlation?

- Regression and correlation are closely related. Correlation measures the strength of the *linear association* between two variables: x and y .
- Correlation treats x and y symmetrically:
 - The correlation of x with y is the same as the correlation of y with x . Correlation has no units.
- Correlation is always between -1 and $+1$.
 - Correlation *can* be exactly equal to -1 or $+1$, but these values are unusual in real data because they mean that all the data points fall *exactly* on a single straight line.
 - A correlation near zero corresponds to a weak linear association.

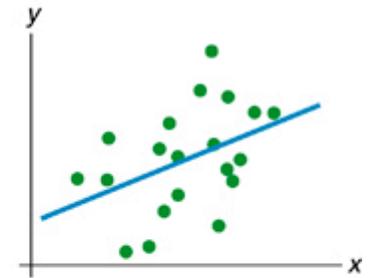
Correlation



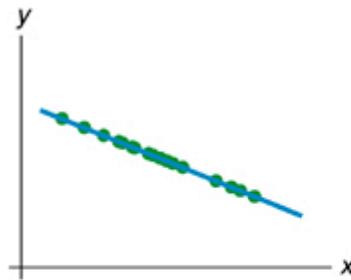
(a) Perfect positive linear correlation
 $r = 1$



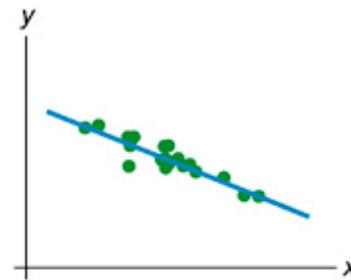
(b) Strong positive linear correlation
 $r = 0.9$



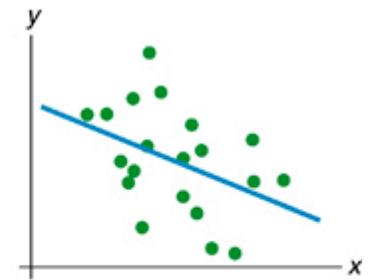
(c) Weak positive linear correlation
 $r = 0.4$



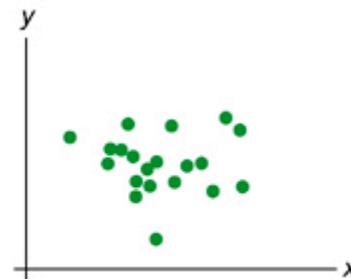
(d) Perfect negative linear correlation
 $r = -1$



(e) Strong negative linear correlation
 $r = -0.9$



(f) Weak negative linear correlation
 $r = -0.4$



(g) No linear correlation
(linearly uncorrelated)
 $r = 0$

How to Use Calculator to Find the Correlation Coefficient, r ?

Here are the steps:

Under STAT EDIT choose “1: Edit,” enter the values for both variables: x and y, under column title L1 and L2.

Now, hit 2nd CATALOG (on the zero key) to turn on the correlation coefficient's calculation feature of your TI-83/84+ calculator. Scroll down until you find DiagnosticOn. Hit ENTER twice. It should say Done. (From now on, your calculator will be able to find correlation, unless the battery is dead.)

Under STAT CALC choose “4: LinReg(ax+b),” enter “L1,L2” and hit ENTER. (You now see not only “ r ,” but also the linear regression line $y=ax+b$.)

Please go through the Calculator Tutorial on this topic.

What are Interpolation and Extrapolation?

The U.S. sales of Toyota vehicles in millions is listed below.

Year	1998	2000	2002
Vehicles	1.4	1.6	1.8

Writing the equation of the line passing through these three points yields the following equation which models the data exactly.

$$y = .1x - 198.4$$

Example of Interpolation: Using the model to predict the sales in the year 1999 we have $y = .1(1999) - 198.4 = 1.5$. This is an example of interpolation because 1999 lies between 1998 and 2002.

Example of Extrapolation: Using the model to predict the sales in the year 2004 we have $y = .1(2004) - 198.4 = 2$. This is an example of extrapolation because 2004 does not lie between 1998 and 2002.

What have we learned?

We have learned to:

1. Recognize linear equations.
2. Solve linear equations symbolically and graphically.
3. Find the zeros of a function.
4. Identify solutions, zeros, and x-intercept.
5. Solve an equation for a specified variable.

What have we learned? (Cont.)

6. Identify a table of values for a linear function.
7. Use constant first differences.
8. Model data with a linear function.
9. Use linear regression to model data.
10. Apply problem-solving strategies.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition
- Weiss, Neil A., Introductory Statistics, 8th Edition