MAC 1105

Module 3

System of Equations and Inequalities

Learning Objectives

Upon completing this module, you should be able to:

1. Evaluate functions of two variables.
2. Apply the method of substitution.
3. Apply the elimination method.
4. Solve system of equations symbolically.
5. Apply graphical and numerical methods to system of equations.
6. Recognize different types of linear systems.

Learning Objectives (Cont.)

7. Use basic terminology related to inequalities.
8. Use interval notation.
9. Solve linear inequalities symbolically.
10. Solve linear inequalities graphically and numerically.
11. Solve double inequalities.
12. Graph a system of linear inequalities.

System of Equations and Inequalities

There are two major topics in this module:

- System of Linear Equations in Two Variables
- Solutions of Linear Inequalities

Do We Really Use Functions of Two Variables?

- The answer is YES.
- Many quantities in everyday life depend on more than one variable.

Examples

- Area of a rectangle requires both width and length.
- Heat index is the function of temperature and humidity.
- Wind chill is determined by calculating the temperature and wind speed.
- Grade point average is computed using grades and credit hours.

Let's Take a Look at the Arithmetic Operations

- The arithmetic operations of addition, subtraction, multiplication, and division are computed by *functions of two inputs*.
- The addition function of f can be represented symbolically by $f(x,y) = x + y$, where $z = f(x,y)$.
  - The *independent variables* are x and y.
  - The *dependent variable* is z. The z output depends on the inputs x and y.

Here are Some Examples

For each function, evaluate the expression and interpret the result.

a) $f(5, -2)$ where $f(x,y) = xy$

b) $A(6,9)$, where $A(b,h) = \frac{1}{2}bh$ calculates the area of a triangle with a base of 6 inches and a height of 9 inches.

Solution

- $f(5, -2) = (5)(-2) = -10$.
- $A(6,9) = \frac{1}{2}(6)(9) = 27$

  If a triangle has a base of 6 inches and a height of 9 inches, the area of the triangle is 27 square inches.

What is a System of Linear Equations?

- A *linear equation in two variables* can be written in the form $ax + by = k$, where a, b, and k are constants, and a and b are not equal to 0.
- A pair of equations is called a *system of linear equations* because they involve solving more than one linear equation at once.
- A *solution* to a system of equations consists of an x-value and a y-value that satisfy both equations simultaneously.
- The set of all solutions is called the *solution set*.

How to Use the Method of Substitution to solve a system of two equations?

THE METHOD OF SUBSTITUTION

To use the method of substitution to solve a system of two equations in two variables, perform the following steps.

STEP 1: Choose a variable in one of the two equations. Solve the equation for that variable.

STEP 2: Substitute the result from **STEP 1** into the other equation and solve for the remaining variable.

STEP 3: Use the value of the variable from **STEP 2** to determine the value of the other variable. To do this, you may want to use the equation you found in **STEP 1**.

Note: To check your answer, substitute the value of each variable into the *given* equations. These values should satisfy *both* equations.

How to Solve the System Symbolically?

Solve the system symbolically.

$$4x + 2y = 8$$
$$3x - 7y = -11$$

Solution

Step 1: Solve one of the equations for one of the variables.

$$4x + 2y = 8$$
$$2y = -4x + 8$$
$$y = -2x + 4$$

Step 2: Substitute $-2x + 4$ for *y* in the second equation.

$$3x - 7(-2x + 4) = -11$$
$$3x + 14x - 28 = -11$$
$$x = 1$$

How to Solve the System Symbolically? (Cont.)

Step 3: Substitute x = 1 into the equation $y = -2x + 4$ from **Step 1**. We find that

$$y = 2.$$

Check:

- $3(1) - 7(2) = -11 \qquad 4(1) + 2(2) = 8$
- The ordered pair is (1, 2) since the solutions check in *both* equations.

Example with Infinitely Many Solutions

Solve the system.

$$8x - 2y = -4$$
$$-4x + y = 2$$

- Solution
- Solve the second equation for *y*.

$$-4x + y = 2$$
$$y = 4x + 2$$

- Substitute 4*x* + 2 for *y* in the first equation, solving for *x*.

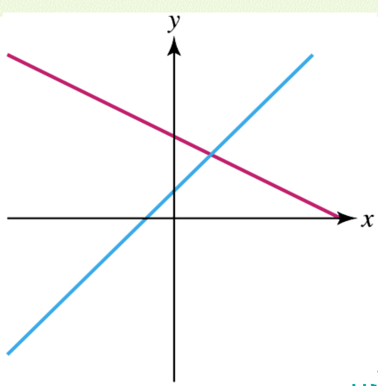
$$8x - 2(4x + 2) = -4$$
$$8x - 8x - 4 = -4$$
$$-4 = -4$$

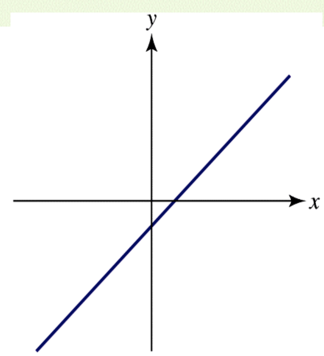
- The equation –4 = –4 is an identity that is always true and indicates that there are infinitely many solutions. The two equations are equivalent.

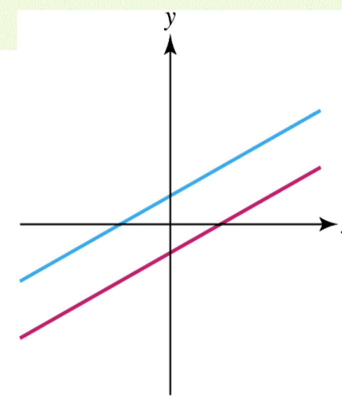
Possible Graphs of a System of Two Linear Equations in Two Variables

POSSIBLE GRAPHS OF A SYSTEM OF TWO LINEAR EQUATIONS IN TWO VARIABLES

1. The graphs of the two equations are distinct lines that intersect at one point. The system is *consistent*. There is one solution, which is given by the coordinates of the point of intersection. In this case the equations are *independent*.
2. The graphs of the two equations are the same line. The system is *consistent*. There are infinitely many solutions, and the equations are *dependent*.
3. The graphs of the two equations are distinct parallel lines. The system is *inconsistent*. There are no solutions.







How to Use Elimination Method to Solve System of Equations?

Use elimination to solve each system of equations, if possible. Identify the system as consistent or inconsistent. If the system is consistent, state whether the equations are dependent or independent. Support your results graphically.

a) $3x - y = 7$
$5x + y = 9$

b) $5x - y = 8$
$-5x + y = -8$

c) $x - y = 5$
$x - y = -2$

How to Use Elimination Method to Solve System of Equations? (Cont.)

Solution

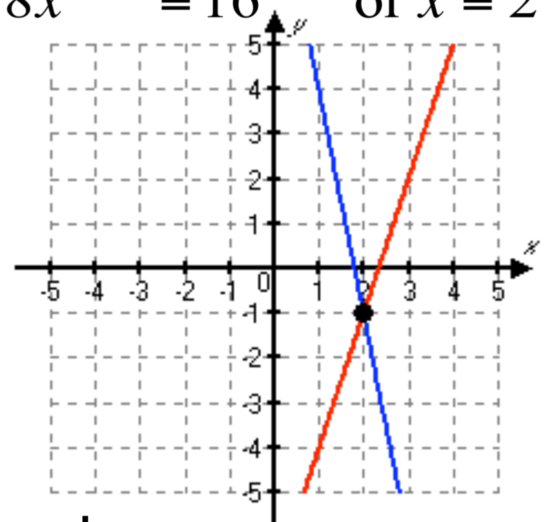
a) $$\begin{aligned} 3x - y &= 7 \\ 5x + y &= 9 \end{aligned}$$

Eliminate *y* by adding the equations.

$$\begin{array}{l} 3x - y = 7 \\ \underline{5x + y = 9} \\ 8x \qquad = 16 \quad \text{or } x = 2 \end{array}$$

Find *y* by substituting $x = 2$ in either equation.

$$\begin{aligned} 3x - y &= 7 \\ 3(2) - y &= 7 \\ -y &= 1 \\ y &= -1 \end{aligned}$$



The solution is (2, –1). The system is consistent and the equations are independent.

How to Use Elimination Method to Solve System of Equations? (Cont.)

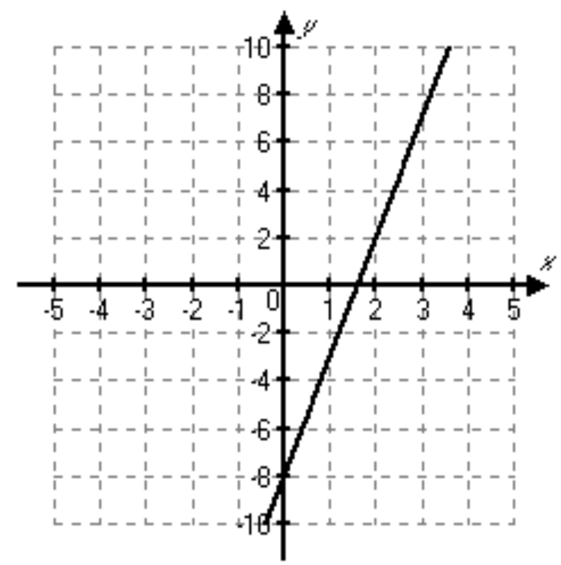
b) $$\begin{aligned} 5x - y &= 8 \\ -5x + y &= -8 \end{aligned}$$

If we add the equations we obtain the following result.

$$\begin{array}{r} 5x - y = 8 \\ \underline{-5x + y = -8} \\ 0 = 0 \end{array}$$

The equation 0 = 0 is an identity that is always true.
The two equations are equivalent.
There are infinitely many solutions.

$\{(x, y) \mid 5x - y = 8\}$



How to Use Elimination Method to Solve System of Equations? (Cont.)

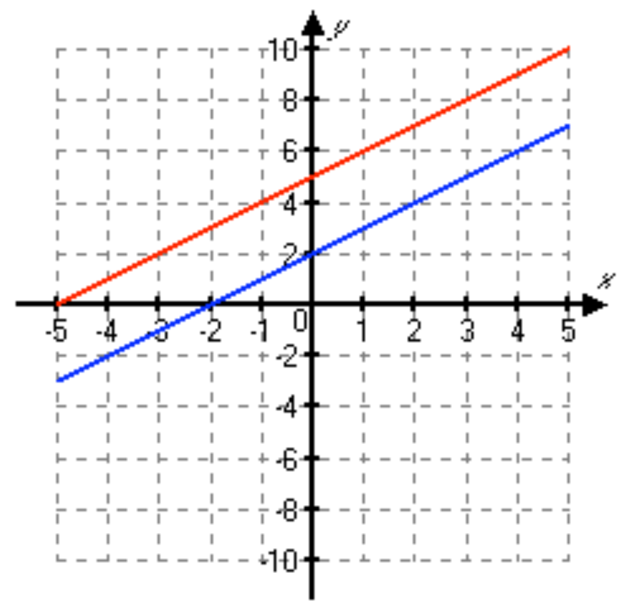
c) $x - y = 5$

$x - y = -2$

If we subtract the second equation from the first, we obtain the following result.

$$\begin{array}{r} x - y = 5 \\ x - y = -2 \\ \hline 0 = 7 \end{array}$$

The equation 0 = 7 is a contradiction that is never true. Therefore there are no solutions, and the system is inconsistent.



Let's Practice Using Elimination

Solve the system by using elimination.

$$3x - 4y = 1$$
$$2x + 3y = 12$$

Solution

Multiply the first equation by 3 and the second equation by 4. Addition eliminates the *y*-variable.

$$\begin{array}{l} 9x - 12y = 3 \\ \underline{8x + 12y = 48} \\ 17x \quad\quad = 51 \qquad \text{or } x = 3 \end{array}$$

Substituting $x = 3$ in $2x + 3y = 12$ results in

$$2(3) + 3y = 12 \text{ or } y = 2$$

The solution is (3, 2).

Terminology related to Inequalities

- Inequalities result whenever the equals sign in an equation is replaced with any one of the symbols: $\leq$, $\geq$, $<$, $>$
- Examples of inequalities include:
- $2x - 7 > x + 13$
- $x^2 \leq 15 - 21x$
- $xy + 9x < 2x^2$
- $35 > 6$

Linear Inequality in One Variable

- A linear inequality in one variable is an inequality that can be written in the form

  $ax + b > 0$ where $a \neq 0$.

  (The symbol may be replaced by $\leq$, $\geq$, $<$, $>$)

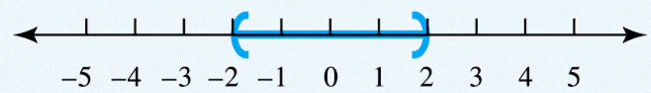
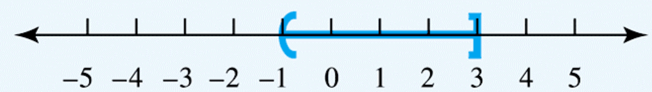
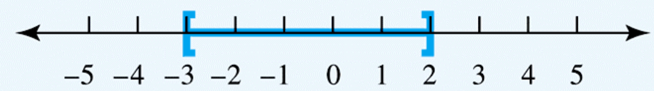
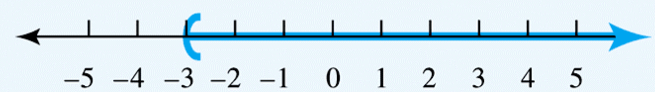
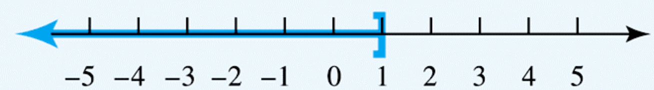
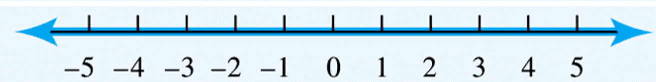
- Examples of linear inequalities in one variable:
  - $5x + 4 \leq 2 + 3x$ simplifies to $2x + 2 \leq 0$
  - $-1(x - 3) + 4(2x + 1) > 5$ simplifies to $7x + 2 > 0$

- Examples of inequalities in one variable which are not linear:
  - $x^2 < 1$

Let's Look at Interval Notation

The solution to a linear inequality in one variable is typically an interval on the real number line. See examples of interval notation below.

TABLE 2.12 Interval Notation

Inequality	Interval Notation	Graph
$-2 < x < 2$	$(-2, 2)$ open interval	
$-1 < x \leq 3$	$(-1, 3]$ half-open interval	
$-3 \leq x \leq 2$	$[-3, 2]$ closed interval	
$x > -3$	$(-3, \infty)$ infinite interval	
$x \leq 1$	$(-\infty, 1]$ infinite interval	
$-\infty < x < \infty$ (entire number line)	$(-\infty, \infty)$ infinite interval	

Multiplied by a Negative Number

Note that $3 < 5$, but if both sides are multiplied by – 1, in order to produce a true statement the > symbol must be used.

$$3 < 5$$

but

$$-3 > -5$$

So when both sides of an inequality are multiplied (or divided) by a negative number the direction of the inequality must be reversed.

How to Solve Linear Inequalities Symbolically?

The procedure for solving a linear inequality symbolically is the same as the procedure for solving a linear equation, except when both sides of an inequality are multiplied (or divided) by a negative number the direction of the inequality is reversed.

Example of Solving a Linear **Equation** Symbolically

Solve $-2x + 1 = x - 2$

$$-2x - x = -2 - 1$$

$$-3x = -3$$

$$x = 1$$

Example of Solving a Linear **Inequality** Symboliclly

Solve $-2x + 1 < x - 2$

$$-2x - x < -2 - 1$$

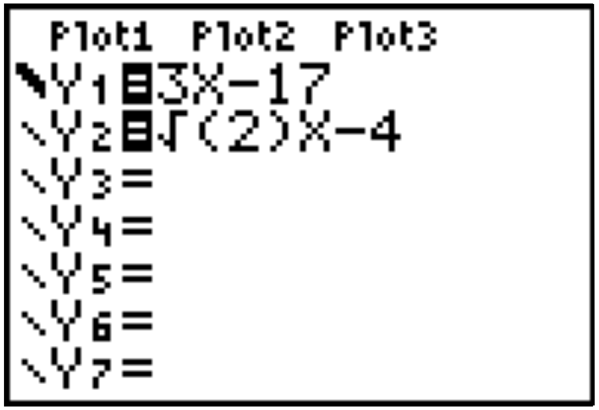
$$-3x < -3$$

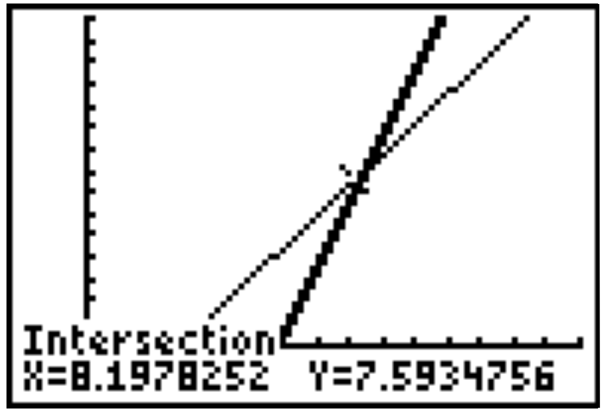
$$x > 1$$

Note that we divided both sides by −3 so the direction of the inequality was reversed. In interval notation the solution set is $(1,\infty)$.

How to Solve a Linear Inequality Graphically?

Solve $3x - 17 > \sqrt{2}x - 4$



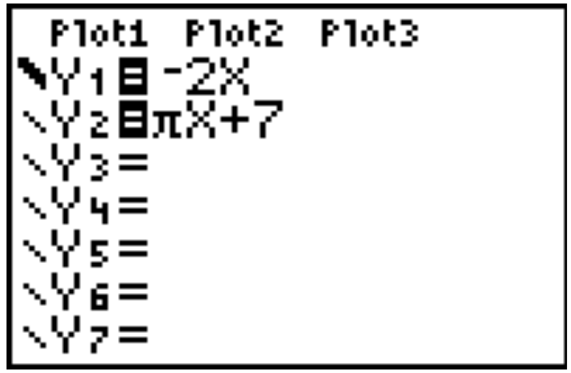


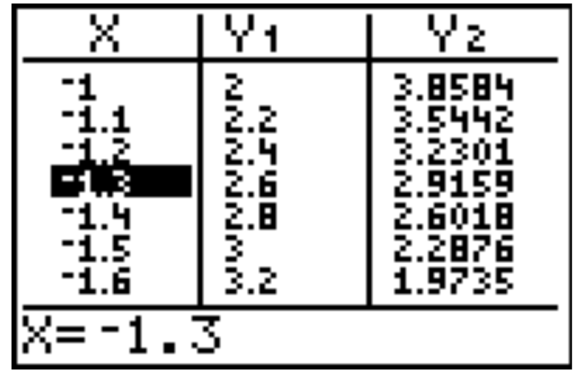
Note that the graphs intersect at the point (8.20, 7.59). The graph of y_1 is above the graph of y_2 to the right of the point of intersection or when $x > 8.20$. Thus, in interval notation, the solution set is $(8.20, \infty)$

How to Solve a Linear Inequality Numerically?

Solve

$$-2x \geq \pi x + 7$$





Note that the inequality above becomes $y_1 \geq y_2$ since we let y_1 equal the left-hand side and y_2 equal the right hand side.

To write the solution set of the inequality we are looking for the values of x in the table for which y_1 is the same or larger than y_2. Note that when $x = -1.3$, y_1 is less than y_2; but when $x = -1.4$, y_1 is larger than y_2. By the Intermediate Value Property, there is a value of x between – 1.4 and – 1.3 such that $y_1 = y_2$. In order to find an approximation of this value, make a new table in which x is incremented by .01 (note that x is incremented by .1 in the table to the left here.)

How to Solve a Linear Inequality Numerically? (cont.)

Solve

$$-2x \geq ðx + 7$$

X	Y1	Y2
-1.34	2.68	2.7903
-1.35	2.7	2.7588
-1.36	2.72	2.7274
-1.37	2.74	2.696
-1.38	2.76	2.6646
-1.39	2.78	2.6332
-1.4	2.8	2.6018

X= -1.36

To write the solution set of the inequality we are looking for the values of x in the table for which y_1 is the same as or larger than y_2. Note that when x is approximately –1.36, y_1 equals y_2 and when x is smaller than –1.36 y_1 is larger than y_2 , so the solutions can be written

$x \leq -1.36$ or $(-\infty, -1.36]$ in interval notation.

How to Solve Double Inequalities?

- Example: Suppose the Fahrenheit temperature x miles above the ground level is given by $T(x) = 88 - 32x$. Determine the altitudes where the air temp is from 30^0 to 40^0.

- We must solve the inequality

  $30 < 88 - 32x < 40$

To solve: Isolate the variable x in the middle of the three-part inequality

How to Solve Double Inequalities? (Cont.)

$$30 < 88 - 32x < 40$$

$$30 - 88 < -32x < 40 - 88$$

$$-58 < -32x < -48$$

$$\frac{-58}{-32} > x > \frac{-48}{-32}$$

$$\frac{29}{16} > x > \frac{3}{2}$$

$$1.8125 > x > 1.5$$

$$1.5 < x < 1.8125$$

Direction reversed –Divided each side of an inequality by a negative

Thus, between 1.5 and 1.8215 miles above ground level, the air temperature is between 30 and 40 degrees Fahrenheit.

How to Graph a System of Linear Inequalities?

The graph of a linear inequality is a half-plane, which may include the boundary. The boundary line is included when the inequality includes a less than or equal to or greater than or equal to symbol.

To determine which part of the plane to shade, select a **test point**.

How to Graph a System of Linear Inequalities? (Cont.)

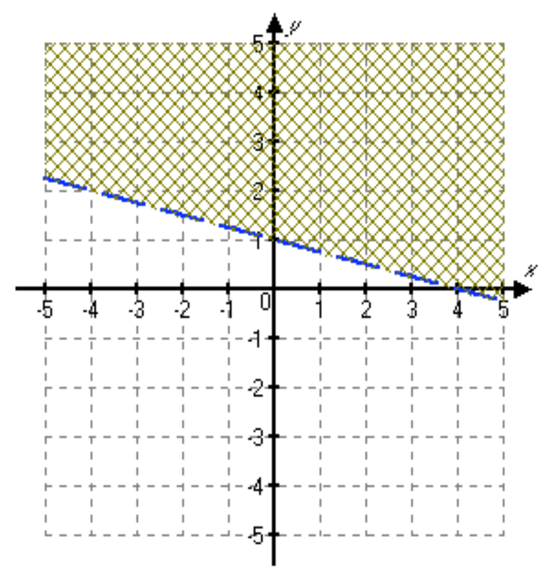
Graph the solution set to the inequality $x + 4y > 4$.

Solution

Graph the line $x + 4y = 4$ using a dashed line.

Use a test point to determine which half of the plane to shade.

Test Point	$x + 4y > 4$	True or False?
(4, 2)	$4 + 4(2) > 4$	True
(0, 0)	$0 + 4(0) > 4$	False



Example

Solve the system of inequalities by shading the solution set. Use the graph to identify one solution.

$x + y \le 3$

$2x + y \ge 4$

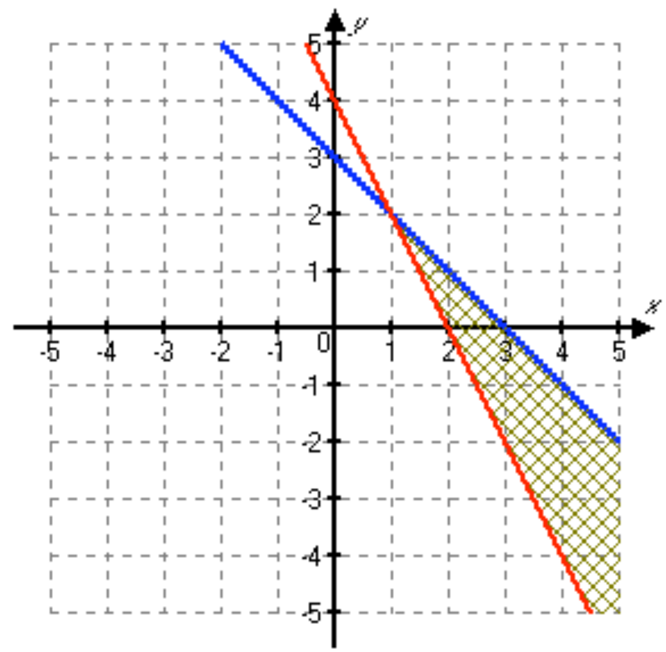
Solution

Solve each inequality for y.

$y \le -x + 3$ (shade below line)

$y \ge -2x + 4$ (shade above line)

The point (4, –2) is a solution.



What have we learned?

We have learned to:

1. Evaluate functions of two variables.
2. Apply the method of substitution.
3. Apply the elimination method.
4. Solve system of equations symbolically.
5. Apply graphical and numerical methods to system of equations.
6. Recognize different types of linear systems.

What have we learned? (Cont.)

7. Use basic terminology related to inequalities.
8. Use interval notation.
9. Solve linear inequalities symbolically.
10. Solve linear inequalities graphically and numerically.
11. Solve double inequalities.
12. Graph a system of linear inequalities.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition