MAC 1105

Module 4
Quadratic Functions and Equations

Learning Objectives

Upon completing this module, you should be able to:

1. Understand basic concepts about quadratic functions and their graphs.
2. Complete the square and apply the vertex formula.
3. Graph a quadratic function by hand.
4. Solve applications and model data.
5. Understand basic concepts about quadratic equations.
6. Use factoring, the square root property, completing the square, and the quadratic formula to solve quadratic equations.
7. Understand the discriminant.
8. Solve problems involving quadratic equations.

Quadratic Functions and Equations

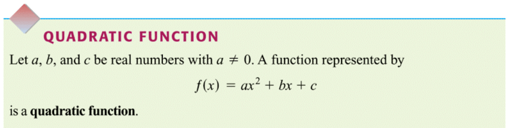
There are two major topics in this module:

- Quadratic Functions
- Solving Quadratic Equations

What is a Quadratic Function?

Recall that a linear function can be written as $f(x) = ax + b$ (or $f(x) = mx + b$). The formula for a quadratic function is different from that of a linear function because it contains an x^2 term.

$f(x) = 3x^2 + 3x + 5 \qquad g(x) = 5 - x^2$



QUADRATIC FUNCTION

Let a, b, and c be real numbers with $a \neq 0$. A function represented by

$$f(x) = ax^2 + bx + c$$

is a **quadratic function**.

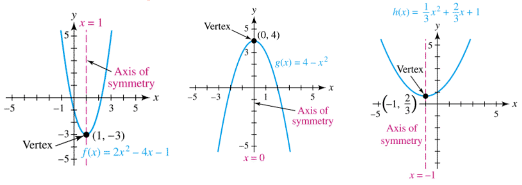
Quadratic Functions (Cont.)

- The graph of a quadratic function is a **parabola**—a U shaped graph that opens either upward or downward.
- A parabola opens upward if a is positive and opens downward if a is negative.

Quadratic Functions (Cont.)

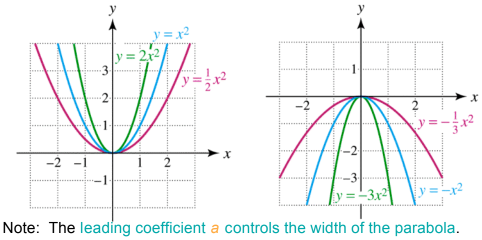
- The highest point on a parabola that opens downward and the lowest point on a parabola that opens upward is called the **vertex**.
- The vertical line passing through the vertex is called the **axis of symmetry.**
- The leading coefficient a controls the width of the parabola. Larger values of $|a|$ result in a narrower parabola, and smaller values of $|a|$ result in a wider parabola.

Example of Different Parabolas



Note: A parabola opens upward if *a* is positive and opens downward if *a* is negative.

Example of Different Parabolas (Cont.)



Note: The leading coefficient *a* controls the width of the parabola. Larger values of |*a*| result in a narrower parabola, and smaller values of |*a*| result in a wider parabola.

Graph of the Quadratic Function

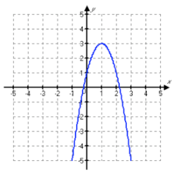
Now, let's use the graph of the quadratic function shown to determine the sign of the leading coefficient, its vertex, and the equation of the axis of symmetry.

Leading coefficient: The graph opens downward, so the leading coefficient *a* is negative.

Vertex: The vertex is the highest point on the graph and is located at (1, 3).

Axis of symmetry: Vertical line through the vertex with equation

x = 1.



Let's Look at the Vertex Form of a Quadratic Function

VERTEX FORM

The parabolic graph of $f(x) = a(x - h)^2 + k$ with $a \neq 0$ has vertex (h, k). Its graph opens upward when $a > 0$ and opens downward when $a < 0$.

We can write the formula $f(x) = x^2 + 10x + 23$ in vertex form by completing the square.

Let's Write the Vertex Form of a Quadratic Function by Completing the Square

$$y = x^2 + 10x + 23$$

$$y - 23 = x^2 + 10x$$

Subtract 23 from each side.

$$y - 23 + 25 = x^2 + 10x + 25$$

Let k = 10; add $(10/2)^2$ = 25.

$$y + 2 = (x + 5)^2$$

Factor perfect square trinomial.

$$y = (x + 5)^2 - 2$$

Vertex Form

How to Use Vertex Formula to Write a Quadratic Function in Vertex Form?

Use the vertex formula to write $f(x) = -3x^2 - 3x + 1$ in vertex form.

Solution: Since a = -3, b = -3 and c = 1, we just need to substitute them into the vertex formula. Mainly, you need to know the vertex formula for *x*; once you have solved for *x*, you can solve for *y*.

1. Begin by finding the vertex.

$$x = -\frac{b}{2a}$$

$$= -\frac{(-3)}{2(-3)}$$

$$= -\frac{1}{2}$$

2. Find *y*.

$$f\left(-\frac{1}{2}\right) = -3\left(\frac{-1}{2}\right)^2 - 3\left(\frac{-1}{2}\right) + 1 = \frac{7}{4}$$

The vertex is: $\left(-\frac{1}{2}, \frac{7}{4}\right)$

Vertex form: $f(x) = -3\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}$

Example

Graph the quadratic equation $g(x) = -3x^2 + 24x - 49$.

Solution

The formula is not in vertex form, but we can find the vertex.

$$x = -\frac{b}{2a} = -\frac{24}{2(-3)} = 4$$

The y-coordinate of the vertex is:

$$g(4) = -3(4)^2 + 24(4) - 49 = -1$$

The vertex is at (4, −1). The axis of symmetry is $x = 4$, and the parabola opens downward because the leading coefficient is negative.

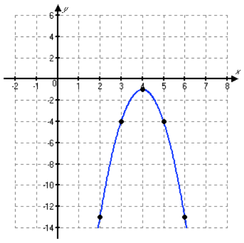
Example (Cont.)

Graph: $g(x) = -3x^2 + 24x - 49$

Table of Values

x	y
2	−13
3	−4
4	−1
5	−4
6	−13

vertex

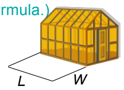


Example of Application

A junior horticulture class decides to enclose a rectangular garden, using a side of the greenhouse as one side of the rectangle. If the class has 32 feet of fence, find the dimensions of the rectangle that give the maximum area for the garden. (Think about using the vertex formula.)

Solution

Let w be the width and L be the length of the rectangle.

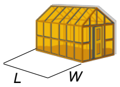


Because the 32-foot fence does not go along the greenhouse, if follows that $W + L + W = 32$ or $L = 32 - 2W$

Example of Application (Cont.)

The area of the garden is the length times the width.

$$A = LW$$
$$= (32 - 2W)W$$
$$= 32W - 2W^2$$



This is a parabola that opens downward, and by the vertex formula, the maximum area occurs when

$$W = -\frac{32}{2(-2)} = 8 \text{ feet}$$

The corresponding length is
$L = 32 - 2W = 32 - 2(8) = 16$ feet.

The dimensions are 8 feet by 16 feet.

Another Example

A model rocket is launched with an initial velocity of v_o = 150 feet per second and leaves the platform with an initial height of h_o = 10 feet.

a) Write a formula $s(t)$ that models the height of the rocket after t seconds.

b) How high is the rocket after 3 seconds?

c) Find the maximum height of the rocket. Support your answer graphically.

Solution

a)
$$s(t) = -16t^2 + v_o t + h_o$$
$$= -16t^2 + 150t + 10$$

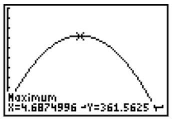
Another Example (Cont.)

b) $s(3) = -16(3)^2 + 150(3) + 10 = 316$

The rocket is 316 feet high after 3 seconds.

c) Because a is negative, the vertex is the highest point on the graph, with a t-coordinate of

$$t = -\frac{b}{2a} = -\frac{150}{2(-16)} = 4.6875 \approx 4.7$$



```
Maximum
X=4.6874996 Y=361.5625
```

The y-coordinate is:

$$s(4.7) = -16(4.7)^2 + 150(4.7) + 10 = 361.56 \text{ feet}$$

The vertex is at (4.7, 361.6).

How to Solve Quadratic Equations?

QUADRATIC EQUATION

A **quadratic equation** in one variable is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$.

The are four basic symbolic strategies in which quadratic equations can be solved.

- Factoring
- Square root property
- Completing the square
- Quadratic formula

Factoring

A common technique used to solve equations that is based on the *zero-product property*.

Example $2x^2 - 4x - 5 = 1$

Solution
$$2x^2 - 4x - 6 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 3 \quad \text{or} \quad x = -1$$

Square Root Property

SQUARE ROOT PROPERTY

Let k be a nonnegative number. Then the solutions to the equation

$$x^2 = k$$

are given by $x = \pm\sqrt{k}$.

Example:

$$-16t^2 + 68 = 0$$
$$-16t^2 = -68$$
$$t^2 = \frac{-68}{-16}$$
$$t^2 = \frac{17}{4}$$
$$t = \pm\sqrt{\frac{17}{4}}$$
$$t \approx \pm 2.1$$

Completing the Square

Completing the square is useful when solving quadratic equations that do not factor easily.

If a quadratic equation can be written in the form where k and d are constants, then the equation can be solved using

$$x^2 + kx + \left(\frac{k}{2}\right)^2 = \left(x + \frac{k}{2}\right)^2.$$

Completing the Square (Cont.)

Solve $2x^2 + 6x = 7$.

Solution

$$2x^2 + 6x = 7$$

$$x^2 + 3x = \frac{7}{2}$$

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{7}{2} + \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{23}{4}$$

$$x + \frac{3}{2} = \pm\sqrt{\frac{23}{4}}$$

$$x = -\frac{3}{2} \pm \sqrt{\frac{23}{4}} = -\frac{3}{2} \pm \frac{\sqrt{23}}{2}$$

Quadratic Formula

QUADRATIC FORMULA

The solutions to the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solve the equation $2x^2 - 5x - 9 = 0.$

Solution

Let $a = 2$, $b = -5$, and $c = -9$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-9)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{97}}{4}$$

Quadratic Equations and the Discriminant

QUADRATIC EQUATIONS AND THE DISCRIMINANT

To determine the number of real solutions to $ax^2 + bx + c = 0$, with $a \neq 0$, evaluate the discriminant $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real solutions.
2. If $b^2 - 4ac = 0$, there is one real solution.
3. If $b^2 - 4ac < 0$, there are no real solutions.

Use the discriminant to determine the number of solutions to the quadratic equation

$$b^2 - 4ac = (-6)^2 - 4(-3)(15) = 216$$

Solution

Since $b^2 - 4ac > 0$, the equation has two real solutions.

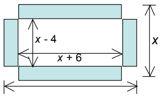
Example

A box is is being constructed by cutting 2 inch squares from the corners of a rectangular sheet of metal that is 10 inches longer than it is wide. If the box has a volume of 238 cubic inches, find the dimensions of the metal sheet.

Solution

Step 1: Let *x* be the width and *x* + 10 be the length.

Step 2: Draw a picture.



Since the height times the width times the length must equal the volume, or 238 cubic inches, the following can be written

---

Example

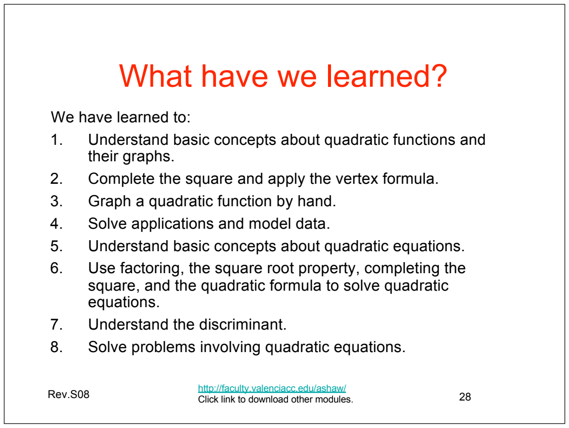
$$2(x-4)(x+6) = 238 \text{ or}$$
$$(x-4)(x+6) = 119$$

Step 3: Write the quadratic equation in the form $ax^2 + bx + c = 0$ and factor.

$$x^2 + 2x - 24 = 119$$
$$x^2 + 2x - 143 = 0$$
$$(x+13)(x-11) = 0$$
$$x = -13 \text{ or } x = 11$$

The dimensions can not be negative, so the width is 11 inches and the length is 10 inches more, or 21 inches.

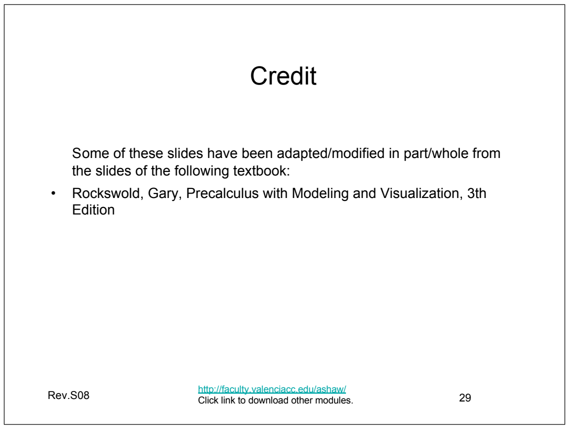
Step 4: After the 2 square inch pieces are cut out, the dimensions of the bottom of the box are 11 – 4 = 7 inches by 21 – 4 = 17 inches. The volume of the box is then 2 x 7 x 17 = 238, which checks.



What have we learned?

We have learned to:

1. Understand basic concepts about quadratic functions and their graphs.
2. Complete the square and apply the vertex formula.
3. Graph a quadratic function by hand.
4. Solve applications and model data.
5. Understand basic concepts about quadratic equations.
6. Use factoring, the square root property, completing the square, and the quadratic formula to solve quadratic equations.
7. Understand the discriminant.
8. Solve problems involving quadratic equations.



Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition