MAC 1105

Module 5
Transformation of Graphs

Learning Objectives

Upon completing this module, you should be able to:

1. Recognize the characteristics common to families of functions.
2. Evaluate and graph piecewise-defined functions.
3. Identify vertical and horizontal asymptotes.
4. Graph functions using vertical and horizontal translations.
5. Graph functions using stretching and shrinking.
6. Graph functions using reflections.
7. Combine transformations.
8. Determine if a function is odd, even, or neither.

Transformation of Graphs

There are two major topics in this module:

- A Library of Functions
- Transformation of Graphs

What are Power Functions?

Power functions typically have rational exponents.
A special type of power function is a *root function*.

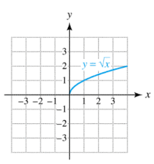
POWER FUNCTION

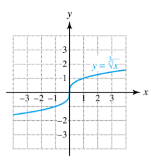
A function f given by $f(x) = x^b$, where b is a constant, is a **power function**. If $b = \frac{1}{n}$ for some integer $n \geq 2$, then f is a **root function** given by $f(x) = x^{1/n}$, or equivalently, $f(x) = \sqrt[n]{x}$.

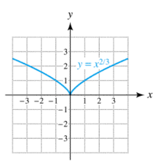
Examples of power functions include:
$f_1(x) = x^2$, $f_2(x) = x^{3/4}$, $f_3(x) = x^{0.4}$, and $f_4(x) = \sqrt[3]{x^2}$

What are Power Functions? (Cont.)

- Often, the domain of a power function f is restricted to nonnegative numbers.
- Suppose the rational number p/q is written in lowest terms. The the domain of $f(x) = x^{p/q}$ is all real numbers whenever q is odd and all nonnegative numbers whenever q is even.
- The following graphs show 3 common power functions.







What are Polynomial Functions?

Polynomial functions are frequently used to approximate data.

POLYNOMIAL FUNCTION

A **polynomial function f of degree n in the variable x** can be represented by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

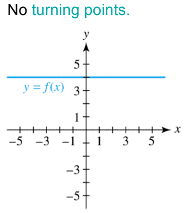
where each coefficient a_k is a real number, $a_n \neq 0$, and n is a nonnegative integer. The **leading coefficient** is a_n and the **degree** is n.

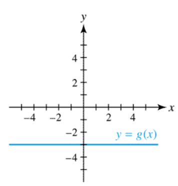
A polynomial function of degree 2 or higher is a nonlinear function.

Next, let’s look at the characteristics of different types of polynomial functions.

Constant Polynomial Functions

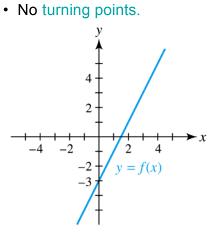
* No *x*-intercepts.
* No turning points.

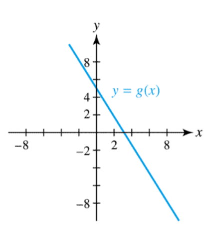




Linear Polynomial Functions

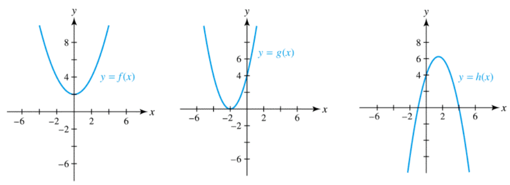
- Degree 1.
- One *x*-intercepts.
- No turning points.





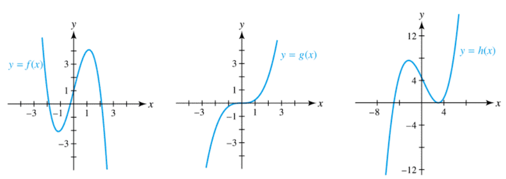
Quadratic Polynomial Functions

- Degree 2 - parabola that opens up or down.
- Zero, one or two *x*-intercepts.
- Exactly ONE turning point, which is also the VERTEX.



Cubic Polynomial Functions

- Degree 3.
- Up to three x-intercepts.
- Up to two turning points.



What is a Piecewise-Defined Function?

A piecewise-defined function is simply a function defined by more than one formula on its domain.

Examples:

- Step function
- Absolute value function

Example of a Piecewise-Defined Function

Undergraduate Classification at Study-Hard University (SHU) is a function of Hours Earned. We can write this in function notation as $C = f(H)$.

–From Catalogue – Verbal Representation

```
No student may be classified as a sophomore until
after earning at least 30 semester hours.

No student may be classified as a junior until after
earning at least 60 hours.

No student may be classified as a senior until after
earning at least 90 hours.
```


Example of a Piecewise-Defined Function (Cont.)

Symbolic Representation

$$C = f(H) = \begin{cases} \text{Freshman if } 0 \le H < 30 \\ \text{Sophomore if } 30 \le H < 60 \\ \text{Junior if } 60 \le H < 90 \\ \text{Senior if } 90 \le H < 120 \end{cases}$$

- Evaluate $f(20)$
  – $f(20)$ = Freshman
- Evaluate $f(30)$
  – $f(30)$ = Sophomore
- Evaluate $f(59)$
  – $f(59)$ = Sophomore
- Evaluate $f(61)$
  – $f(61)$ = Junior
- Evaluate $f(100)$
  – $f(100)$ = Senior

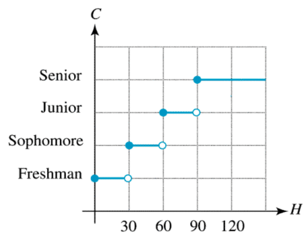
Example of a Piecewise-Defined Function (Cont.)

Symbolic Representation

$$C = f(H) = \begin{cases} \text{Freshman if } 0 \le H < 30 \\ \text{Sophomore if } 30 \le H < 60 \\ \text{Junior if } 60 \le H < 90 \\ \text{Senior if } 90 \le H < 120 \end{cases}$$

- Why is $f(20)$ = Freshman?
  – To evaluate $f(20)$ one must ask: when H has a value of 20, what is the value of C? In other words, what is the classification of a student who has earned 20 credit hours? 20 fits into the category $0 \le H < 30$, so the answer is Freshman.

The Graphical Representation



Example

Evaluate $f(x)$ at −6, 0, and 4.

$$f(x)\begin{cases} -5x & \text{if } x < -5 \\ x^3+1 & \text{if } -4 < x \le 2 \\ 3-x^2 & \text{if } x > 2 \end{cases}$$

Solution

To evaluate $f(-6)$ we use the formula $-5x$ because −6 is < −5.

$f(-6) = -5(-6) = 30$

Similarly, $f(0) = x^3 + 1 = (0)^3 + 1 = 1$

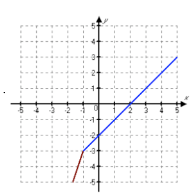
$f(4) = 3 - x^2 = 3 - (4)^2 = -13$

Another Example of a Piecewise-Defined Function

Graph the following function and evaluate $f(-5)$, $f(-1)$, $f(0)$

$$f(x) = \begin{cases} 3x & \text{if } x < -1 \\ x-2 & \text{if } x \ge -1 \end{cases}$$

- First graph the line $y = 3x$, but restrict the graph to points which have an x-coordinate < −1.
- Now graph $x - 2$ but restrict the graph to points which have an x-coordinate −1 and larger.
- The resulting graph is:



Another Example of a Piecewise-Defined Function (Cont.)

Graph the following function and evaluate $f(-5)$, $f(-1)$, $f(0)$

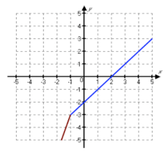
$$f(x) = \begin{cases} 3x & \text{if } x < -1 \\ x - 2 & \text{if } x \ge -1 \end{cases}$$

- Given an input, to see which rule to use to compute the output, check to see into which category the input fits. In other words, is an input less than −1 or greater than or equal to −1?

- Since $-5 < -1$, use the rule $y = 3x$. Thus $f(-5) = 3(-5) = -15$

- Since $-1 \ge -1$, use the rule $y = x - 2$. Thus $f(-1) = -1 - 2 = -3$

- Since $0 \ge -1$, use the rule $y = x - 2$. Thus $f(0) = 0 - 2 = -2$



One More Example

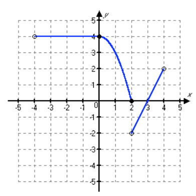
Complete the following.

a) Sketch the graph of *f*.

b) Determine if *f* is continuous on its domain.

c) Evaluate $f(1)$.

$$f(x) \begin{cases} 4 & \text{if } -4 < x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \\ 2x - 6 & \text{if } 2 < x < 4 \end{cases}$$



Solution

a) Graph as shown to the right.

b) The domain is not continuous since there is a break in the graph.

c) $f(1) = 4 - x^2 = 4 - (1)^2 = 3$

What is an Absolute Value Function?

- The symbol for the absolute value of x is $|x|$
- The absolute value function is a piecewise-defined function.
- The output from the absolute value function is never negative.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

$|-5| = 5$

$|-2.3| = 2$

$|0| = 0$

$|5| = 5$

$|2.3| = 2.3$

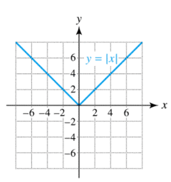


FIGURE 2.63 The Absolute Value Function

What is a Rational Function?

RATIONAL FUNCTION

A function f represented by $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is a **rational function**.

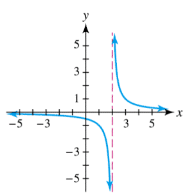
A rational function is a nonlinear function. The domain of a rational function includes all real numbers except the zeros of the denominator *q(x)*.

If *p(x)* = 1 and *q(x)* = *x*, then *f(x)* = 1/*x* is called the reciprocal function. Now, try to use your calculator to plot the reciprocal function.

What is a Vertical Asymptote?

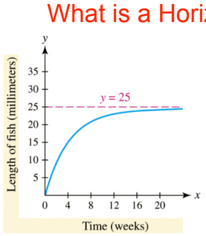
VERTICAL ASYMPTOTE

The line $x = k$ is a **vertical asymptote** of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$, as x approaches k from either the left or the right.



This is a graph of the reciprocal function shifted to the right by 2 units, the line x = 2 is a vertical asymptote.

What is a Horizontal Asymptote?



In this graph, the line y = 25 is a horizontal asymptote.

HORIZONTAL ASYMPTOTE

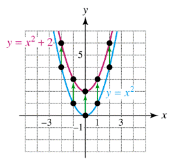
The line $y = b$ is a **horizontal asymptote** of the graph of f, if $f(x) \to b$ as x approaches either ∞ or $-\infty$.

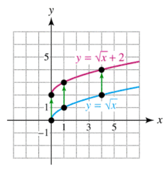
Transformation of Graphs: Vertical Shifts

A graph is shifted up or down. The shape of the graph is not changed—only its position.

x	-2	-1	0	1	2
y	6	3	2	3	6

x	0	1	4
y	2	3	4



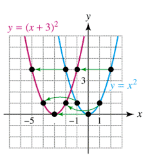


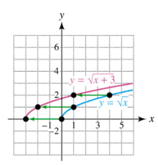
Transformation of Graphs: Horizontal Shifts

A graph is shifted left or right. The shape of the graph is not changed—only its position.

x	-5	-4	-3	-2	-1
y	4	1	0	1	4

x	-3	-2	1
y	0	1	2





Transformation of Graphs

VERTICAL AND HORIZONTAL SHIFTS

Let f be a function, and let k be a positive number.

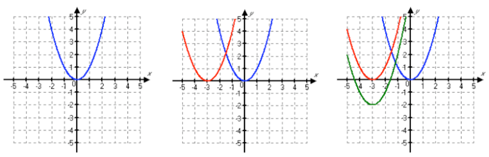
To graph	Shift the graph of $y = f(x)$ by k units
$y = f(x) + k$	upward
$y = f(x) - k$	downward
$y = f(x - k)$	right
$y = f(x + k)$	left

Example of Transformation of Graphs

Shifts can be combined to translate a graph of $y = f(x)$ both vertically and horizontally.

Shift the graph of $y = x^2$ to the left 3 units and downward 2 units.

$y = x^2$ $\quad$ $y = (x + 3)^2$ $\quad$ $y = (x + 3)^2 - 2$



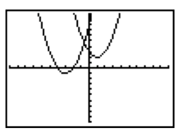
Another Example

Find an equation that shifts the graph of $f(x) = x^2 - 2x + 3$ left 4 units and down 3 units.

Solution

To shift the graph left 4 units, replace x with $(x + 4)$ in the formula for $f(x)$.

$y = f(x + 4) = (x + 4)^2 - 2(x + 4) + 3$



To shift the graph down 3 units, subtract 3 to the formula.

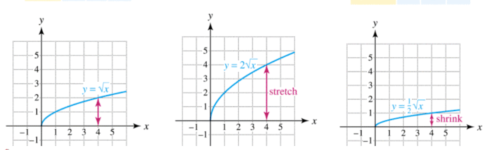
$y = f(x + 4) - 3 = (x + 4)^2 - 2(x + 4) + 3 - 3$

Vertical Stretching and Shrinking

x	0	1	4
$f(x)$	0	1	2

x	0	1	4
$2f(x)$	0	2	4

x	0	1	4
$\frac{1}{2}f(x)$	0	$\frac{1}{2}$	1



VERTICAL STRETCHING AND SHRINKING

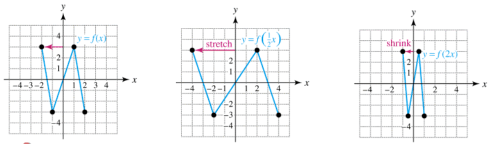
If the point (x, y) lies on the graph of $y = f(x)$, then the point (x, cy) lies on the graph of $y = cf(x)$. If $c > 1$, the graph of $y = cf(x)$ is a vertical stretching of the graph of $y = f(x)$, whereas if $0 < c < 1$, the graph of $y = cf(x)$ is a vertical shrinking of the graph of $y = f(x)$.

Horizontal Stretching and Shrinking

x	-2	-1	1	2
$f(x)$	3	-3	3	-3

x	-4	-2	2	4
$f(\frac{1}{2}x)$	3	-3	3	-3

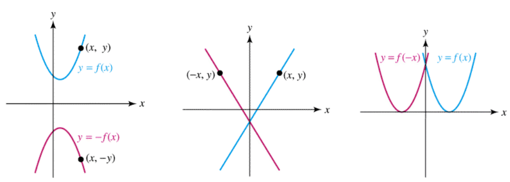
x	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
$f(2x)$	3	-3	3	-3



HORIZONTAL STRETCHING AND SHRINKING

If the point (x, y) lies on the graph of $y = f(x)$, then the point $\left(\frac{x}{c}, y\right)$ lies on the graph of $y = f(cx)$. If $c > 1$, the graph of $y = f(cx)$ is a horizontal shrinking of the graph of $y = f(x)$, whereas if $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretching of the graph of $y = f(x)$.

Reflection of Graphs



REFLECTIONS OF GRAPHS ACROSS THE x- AND y-AXES

1. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ across the x-axis.
2. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ across the y-axis.

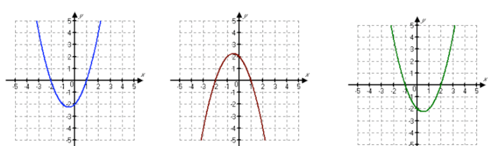
Reflection of Graphs

For the function $f(x) = x^2 + x - 2$ graph its reflection across the x-axis and across the y-axis.

Solution The graph is a parabola with x-intercepts -2 and 1.

To obtain its reflection across the x-axis, graph $y = -f(x)$, or $y = -(x^2 + x - 2)$. The x-intercepts have not changed.

To obtain the reflection across the y-axis let $y = f(-x)$, or $y = (-x)^2 - x - 2$. The x-intercepts have changed to -1 and 2.



Combining Transformations

Transformations of graphs can be combined to create new graphs.

For example the graph of
$y = -3(x + 3)^2 + 1$ can be obtained by performing four transformations on the graph of $y = x^2$.

1. Shift of the graph 3 units left: $y = (x + 3)^2$
2. Vertically stretch the graph by a factor of 3: $y = 3(x + 3)^2$
3. Reflect the graph across the x-axis: $y = -3(x + 3)^2$
4. Shift the graph upward 1 unit: $y = -3(x + 3)^2 + 1$

Combining Transformations (Cont.)

Stretch vertically by a factor of 3

Shift to the left 3 units.

$$y = -3(x + 3)^2 + 1$$

Reflect across the x-axis.

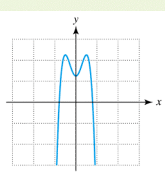
Shift upward 1 unit.

What is an Even Function?

EVEN FUNCTION

A function f is an **even function** if $f(-x) = f(x)$ for every x in its domain. The graph of an even function is symmetric with respect to the y-axis.

If a graph was folded along the y-axis, and the right and left sides would match, the graph would be *symmetric with respect to the y-axis*. A function whose graph satisfies this characteristic is called an **even function**.

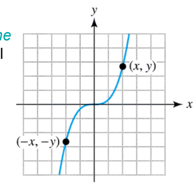


What is an Odd Function?

ODD FUNCTION

A function f is an **odd function** if $f(-x) = -f(x)$ for every x in its domain. The graph of an odd function is symmetric with respect to the origin.

When the symmetry occurs in respect *to the origin*. If the graph could rotate, the original graph would reappear after half a turn (clockwise or counter-clockwise.) This represents an **odd function**.



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Is the Function Even or Odd?

Identify whether the function is even or odd.

$$f(x) = 6x^3 - 9x$$

Solution

Since f is a polynomial containing only odd powers of x, it is an *odd function*. This also can be shown symbolically as follows.

$$\begin{aligned} f(-x) &= 6(-x)^3 - 9(-x) \\ &= -6x^3 + 9x \\ &= -(6x^3 - 9x) \\ &= -f(x) \end{aligned}$$

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What have we learned?

We have learned to:

1. Recognize the characteristics common to families of functions.
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Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition