MAC 1105

Module 6

Composite Functions and Inverse Functions

Learning Objectives

Upon completing this module, you should be able to:

1. Perform arithmetic operations on functions.
2. Perform composition of functions.
3. Calculate inverse operations.
4. Identity one-to-one functions.
5. Use horizontal line test to determine if a graph represents a one-to-one function.
6. Find inverse functions symbolically.
7. Use other representations to find inverse functions.

Composite Functions and Inverse Functions

There are two major topics in this module:

- Combining Functions; Composite Functions
- Inverse Functions

Five Ways of Combining Functions

If *f*(*x*) and *g*(*x*) both exist, the sum, difference, product, quotient and composition of two functions *f* and *g* are defined by

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0$$

$$(f \circ g)(x) = f(g(x))$$

Example of Addition of Functions

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 1$

Find the symbolic representation for the function $f + g$ and use this to evaluate $(f + g)(2)$.

$(f + g)(x) = (x^2 + 2x) + (3x - 1)$

$(f + g)(x) = x^2 + 5x - 1$

$(f + g)(2) = 2^2 + 5(2) - 1 = 13$

or $(f + g)(2) = f(2) + g(2)$

$= 2^2 + 2(2) + 3(2) - 1$

$= 13$

Example of Subtraction of Functions

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 1$

Find the symbolic representation for the function $f - g$ and use this to evaluate

$(f - g)(2)$.

$(f - g)(x) = (x^2 + 2x) - (3x - 1)$

$(f - g)(x) = x^2 - x + 1$

So $(f - g)(2) = 2^2 - 2 + 1 = 3$

Example of Multiplication of Functions

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 1$

Find the symbolic representation for the function *fg* and use this to evaluate $(fg)(2)$

$(fg)(x) = (x^2 + 2x)(3x - 1)$

$(fg)(x) = 3x^3 + 6x^2 - x^2 - 2x$

$(fg)(x) = 3x^3 + 5x^2 - 2x$

So $(fg)(2) = 3(2)^3 + 5(2)^2 - 2(2) = 40$

Example of Division of Functions

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 1$

Find the symbolic representation for the function $\frac{f}{g}$ and use this to evaluate

$$\left(\frac{f}{g}\right)(x) = \frac{x^2 + 2(x)}{3(x) - 1}$$

So $$\left(\frac{f}{g}\right)(2) = \frac{2^2 + 2(2)}{3(2) - 1} = \frac{8}{5}$$

Example of Composition of Functions

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 1$

Find the symbolic representation for the function $f \circ g$ and use this to evaluate $(f \circ g)(2)$

$(f \circ g)(x) = f(g(x)) = f(3x - 1) = (3x - 1)^2 + 2(3x - 1)$

$(f \circ g)(x) = (3x - 1)(3x - 1) + 6x - 2$

$(f \circ g)(x) = 9x^2 - 3x - 3x + 1 + 6x - 2$

$(f \circ g)(x) = 9x^2 - 1$

So $(f \circ g)(2) = 9(2)^2 - 1 = 35$

How to Evaluate Combining of Functions Numerically?

Given numerical representations for *f* and *g* in the table

x	5	6	7	8
$f(x)$	8	7	6	5
$g(x)$	6	5	8	7

Evaluate combinations of *f* and *g* as specified.

x	5	6	7	8
$(f+g)(x)$				
$(f-g)(x)$				
$(fg)(x)$				
$(f/g)(x)$				
$(f \circ g)(x)$				

How to Evaluate Combining of Functions Numerically? (Cont.)

$(f + g)(5) = f(5) + g(5) = 8 + 6 = 14$

$(fg)(5) = f(5) \cdot g(5) = 8 \cdot 6 = 48$

$(f \circ g)(5) = f(g(5)) = f(6) = 7$

x	5	6	7	8
$f(x)$	8	7	6	5
$g(x)$	6	5	8	7

Try to work out the rest of them now.

x	5	6	7	8
$(f + g)(x)$	14			
$(f - g)(x)$				
$(fg)(x)$	48			
$(f/g)(x)$				
$(f \circ g)(x)$	7			

How to Evaluate Combining of Functions Numerically? (Cont.)

Check your answers:

x	**5**	**6**	**7**	**8**
$(f+g)(x)$	14	12	14	12
$(f-g)(x)$	2	2	–2	–2
$(fg)(x)$	48	35	48	35
$(f/g)(x)$	4/3	7/5	3/4	5/7
$(f \circ g)(x)$	7	8	5	6

How to Evaluate Combining of Functions Graphically?

Use graph of *f* and *g* below to evaluate

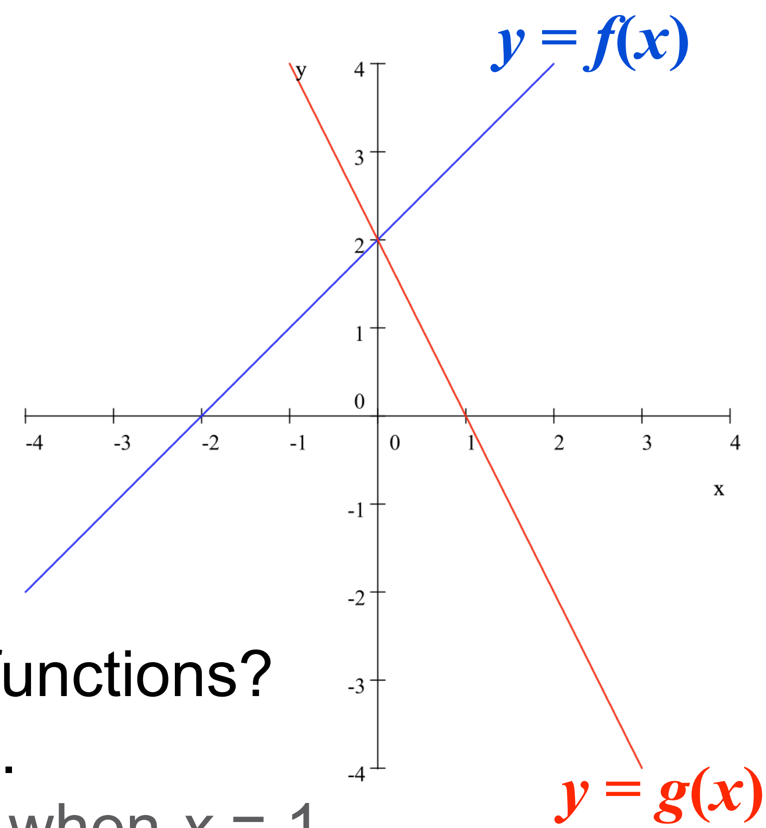
$(f + g)(1)$

$(f - g)(1)$

$(f \cdot g)(1)$

$(f/g)(1)$

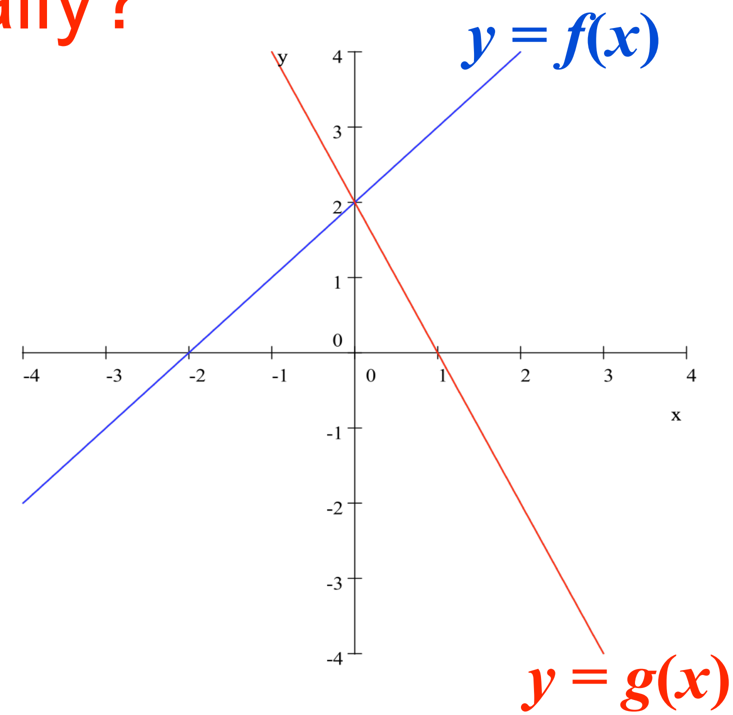
$(f \circ g)(1)$



Can you identify the two functions?
Try to evaluate them now.
Hint: Look at the *y*-value when *x* = 1.

How to Evaluate Combining of Functions Graphically?

Check your answer now.



$(f + g)(1) = f(1) + g(1) = 3 + 0 = 3$

$(f - g)(1) = f(1) - g(1) = 3 - 0 = 3$

$(fg)(1) = f(1) \cdot g(1) = 3 \cdot 0 = 0$

$(f/g)(1)$ is undefined, because division by 0 is undefined.

$(f \circ g)(1) = f(g(1)) = f(0) = 2$

Next, Let's Look at Inverse Functions and Their Representations.

A Quick Review on Function

- $y = f(x)$ means that given an input x, there is just one corresponding output y.
- Graphically, this means that the graph passes the **vertical line test**.
- Numerically, this means that in a table of values for $y = f(x)$ there are **no *x*-values repeated**.

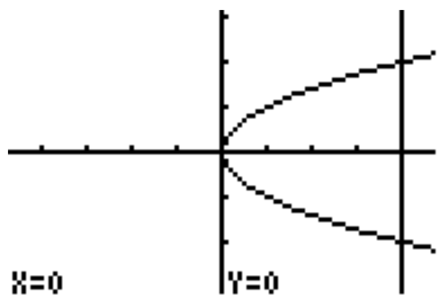
A Quick Example

Given $y^2 = x$, is $y = f(x)$? That is, is y a function of x?

No, because if $x = 4$, y could be 2 or –2.

Note that the graph fails the vertical line test.

x	*y*
4	**– 2**
1	– 1
0	0
1	1
4	**2**



Note that there is a value of ***x*** in the table for which there are two different values of ***y*** (that is, ***x***-values are repeated.)

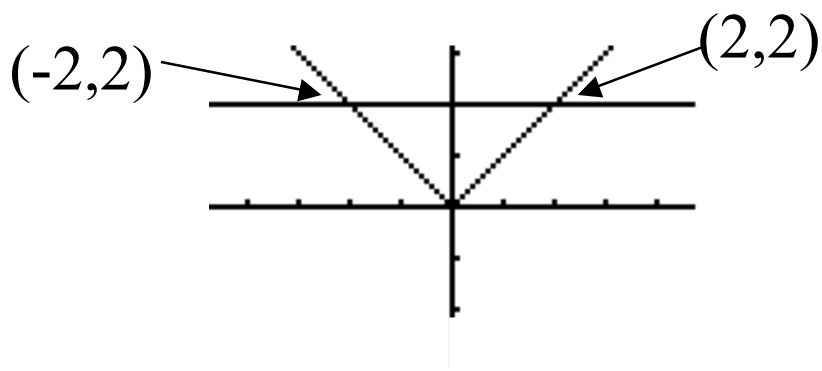
What is One-to-One?

- Given a function $y = f(x)$, f is one-to-one means that given an output y there was just one input x which produced that output.
- Graphically, this means that the graph passes the **horizontal line test**. Every horizontal line intersects the graph at most once.
- Numerically, this means the there are **no *y*-values repeated** in a table of values.

Example

- Given $y = f(x) = |x|$, is f one-to-one?
  - No, because if $y = 2$, x could be 2 or – 2.
    - Note that the graph fails the horizontal line test.

x	***y***
– 2	**2**
– 1	1
0	0
1	1
2	**2**



- Note that there is a value of ***y*** in the table for which there are two different values of ***x*** (that is, *y*-values are repeated.)

What is the Definition of a One-to-One Function?

A function f is a one-to-one function if, for elements c and d in the domain of f,

$$c \neq d \text{ implies } f(c) \neq f(d)$$

Example: Given $y = f(x) = |x|$, f is not one-to-one because $-2 \neq 2$ yet $|-2| = |2|$

What is an Inverse Function?

f^{-1} is a symbol for the inverse of the function f, not to be confused with the reciprocal.

If $f^{-1}(x)$ does NOT mean $1/f(x)$, what does it mean?

$y = f^{-1}(x)$ means that $x = f(y)$

Note that $y = f^{-1}(x)$ is pronounced "*y* equals *f* inverse of *x*."

Example of an Inverse Function

Let *F* be Fahrenheit temperature and let *C* be Centigrade temperature.

$$F = f(C) = (9/5)C + 32$$

$$C = f^{-1}(F) = ?????$$

- The function *f* multiplies an input *C* by 9/5 and adds 32.
- To undo multiplying by 9/5 and adding 32, one should subtract 32 and divide by 9/5

So $C = f^{-1}(F) = (F - 32)/(9/5)$

$$= (5/9)(F - 32)$$

Example of an Inverse Function (Cont.)

$F = f(C) = (9/5)C + 32$

$C = f^{-1}(F) = (5/9)(F - 32)$

- Evaluate $f(0)$ and interpret.

  $f(0) = (9/5)(0) + 32 = 32$

  When the Centigrade temperature is 0, the Fahrenheit temperature is 32.

- Evaluate $f^{-1}(32)$ and interpret.

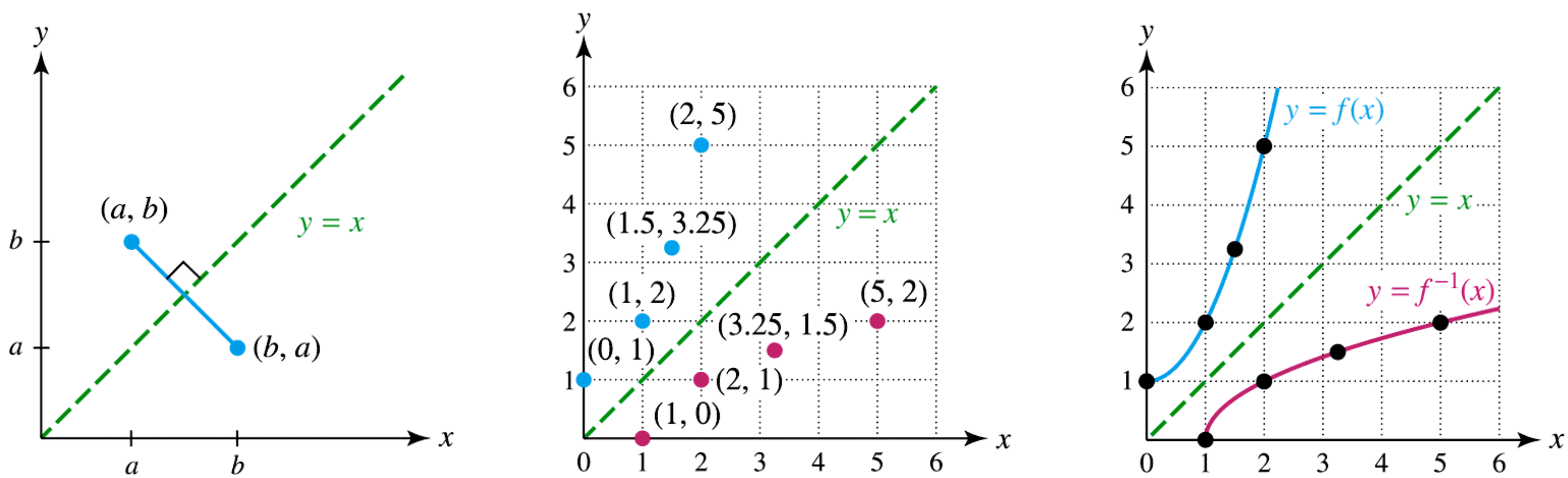
  $f^{-1}(32) = (5/9)(32 - 32) = 0$

  When the Fahrenheit temperature is 32, the Centigrade temperature is 0.

Note that $f(0) = \mathbf{32}$ and $f^{-1}(\mathbf{32}) = 0$

Graph of Functions and Their Inverses

- The graph of f^{-1} is a reflection of the graph of f across the line $y = x$



Note that the domain of f equals the range of f^{-1} and the range of f equals the domain of f^{-1} .

How to Find Inverse Function Symbolically?

- Check that f is a one-to-one function. If not, f^{-1} does not exist.
- Solve the equation $y = f(x)$ for x, resulting in the equation $x = f^{-1}(y)$
- Interchange x and y to obtain $y = f^{-1}(x)$

Example.

- Step 1 - Is this a one-to-one function? Yes. $f(x) = 3x + 2$
- Step 2 - Replace $f(x)$ with y: $y = 3x + 2$
- Step 3 - Solve for x: $3x = y – 2$
- $x = (y – 2)/3$
- Step 4 - Interchange x and y: $y = (x – 2)/3$
- Step 5 - Replace y with $f^{-1}(x)$: So $f^{-1}(x) = (x – 2)/3$

How to Evaluate Inverse Function Numerically?

x	$f(x)$
1	–5
2	–3
3	0
4	3
5	5

The function is one-to-one, so f^{-1} exists.

$f^{-1}(-5) = 1$

$f^{-1}(-3) = 2$

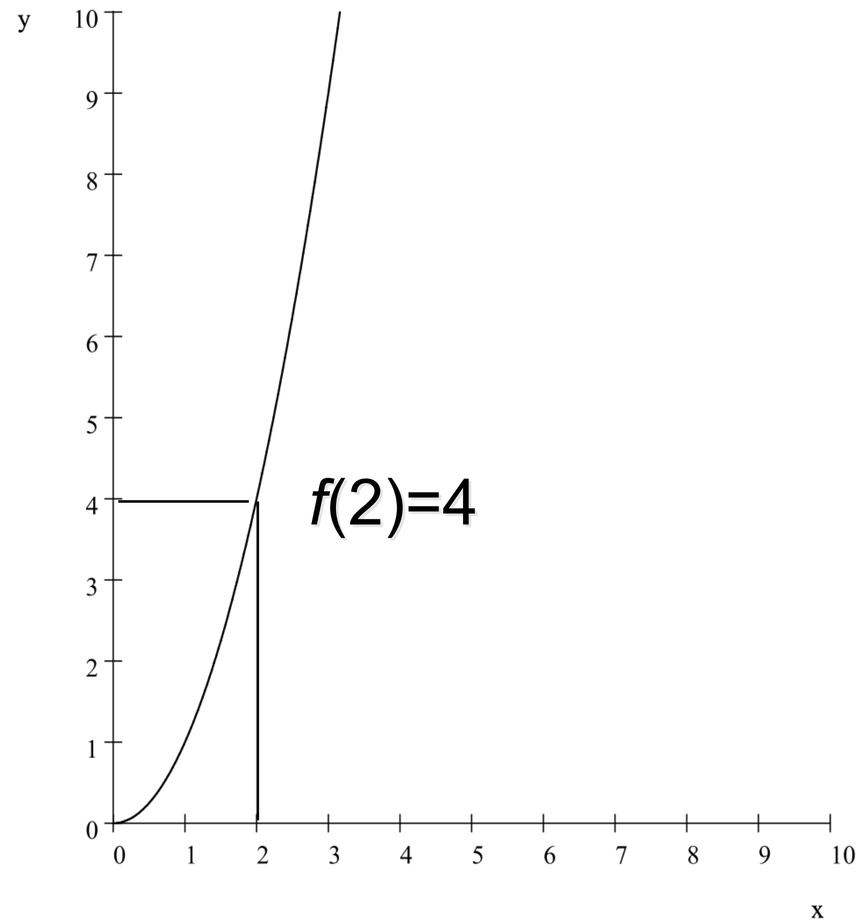
$f^{-1}(0) = 3$

$f^{-1}(3) = 4$

$f^{-1}(5) = 5$

How to Evaluate Inverse Function Graphically?

- The graph of *f* below passes the horizontal line test so *f* is one-to-one.
- Evaluate $f^{-1}(4)$.
- Since the point (2,4) is on the graph of *f*, the point (4,2) will be on the graph of f^{-1} and thus $f^{-1}(4) = 2$



What is the Formal Definition of Inverse Functions?

Let f be a one-to-one function. Then f^{-1} is the inverse function of f, if

- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for every x in the domain of f
- $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

What have we learned?

We have learned to:

1. Perform arithmetic operations on functions.
2. Perform composition of functions.
3. Calculate inverse operations.
4. Identity one-to-one functions.
5. Use horizontal line test to determine if a graph represents a one-to-one function.
6. Find inverse functions symbolically.
7. Use other representations to find inverse functions.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition