MAC 1105

Module 7
Additional Equations and Inequalities

Learning Objectives

Upon completing this module, you should be able to:

1. Use properties of rational exponents (rational powers).
2. Understand radical notation.
3. Solve equations involving rational powers.
4. Solve radical equations.
5. Solve absolute value equations
6. Solve inequalities involving absolute values.
7. Solve quadratic inequalities.

Let's Review Some Properties of Rational Exponents

PROPERTIES OF RATIONAL EXPONENTS

Let m and n be positive integers with $\frac{m}{n}$ in lowest terms and $n \geq 2$. Let r and p be rational numbers. Assume that b is a nonzero real number and that each expression is a real number.

Property	Example
1. $b^{m/n} = (b^m)^{1/n} = (b^{1/n})^m$	$4^{3/2} = (4^3)^{1/2} = (4^{1/2})^3 = 2^3 = 8$
2. $b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$	$8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4$
3. $(b^r)^p = b^{rp}$	$(2^{3/2})^4 = 2^6 = 64$
4. $b^{-r} = \frac{1}{b^r}$	$4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{2}$
5. $b^r b^p = b^{r+p}$	$3^{5/2} \cdot 3^{3/2} = 3^{(5/2)+(3/2)} = 3^4 = 81$
6. $\frac{b^r}{b^p} = b^{r-p}$	$\frac{5^{5/4}}{5^{3/4}} = 5^{(5/4)-(3/4)} = 5^{1/2}$

Let's Practice Some Simplification

Simplify each expression by hand.

$8^{2/3}$ b) $(–32)^{–4/5}$

Solutions

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$$

$$(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

What are Power Functions?

Power functions typically have rational exponents.

A special type of power function is a *root function*.

POWER FUNCTION

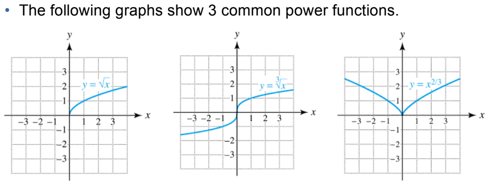
A function f given by $f(x) = x^b$, where b is a constant, is a **power function**. If $b = \frac{1}{n}$ for some integer $n \geq 2$, then f is a **root function** given by $f(x) = x^{1/n}$, or equivalently, $f(x) = \sqrt[n]{x}$.

Examples of power functions include:

$f_1(x) = x^2$, $f_2(x) = x^{3/4}$, $f_3(x) = x^{0.4}$, and $f_4(x) = \sqrt[3]{x^2}$

What are Power Functions? (Cont.)

- Often, the domain of a power function f is restricted to nonnegative numbers.
- Suppose the rational number p/q is written in lowest terms. The domain of $f(x) = x^{p/q}$ is all real numbers whenever q is odd and all nonnegative numbers whenever q is even.
- The following graphs show 3 common power functions.



How to Solve Equations Involving Rational Powers?

Example Solve $4x^{3/2} - 6 = 6$. Approximate the answer to the nearest hundredth, and give graphical support.

Solutions

Symbolic Solution

$4x^{3/2} - 6 = 6$

$4x^{3/2} = 12$

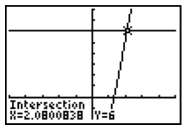
$(x^{3/2})^2 = 3^2$

$x^3 = 9$

$x = 9^{1/3}$

$x = 2.08$

Graphical Solution



How to Solve an Equation Involving Radical?

When solving equations that contain square roots, it is common to square each side of an equation.

Example Solve $\sqrt{3x-2} = x - 2.$

Solution

$$\sqrt{3x-2} = x-2$$
$$\left(\sqrt{3x-2}\right)^2 = (x-2)^2$$
$$3x - 2 = x^2 - 4x + 4$$
$$x^2 - 7x + 6 = 0$$
$$(x-1)(x-6) = 0$$
$$x = 1 \text{ or } x = 6$$

How to Solve an Equation Involving Radical? (Cont.)

Check

$$\sqrt{3(1)-2} = 1-2 \qquad 1 \neq -1$$

$$\sqrt{3(6)-2} = 6-2 \qquad 4 = 4$$

Substituting these values in the original equation shows that the value of 1 is an **extraneous solution** because it does not satisfy the given equation.

Therefore, the only solution is 6.

How to Solve Equation Involving Radicals?

Some equations may contain a cube root.

Solve

Solution $\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}.$

$$\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}$$
$$\left(\sqrt[3]{4x^2 - 4x + 1}\right)^3 = \left(\sqrt[3]{x}\right)^3$$
$$4x^2 - 5x + 1 = 0$$
$$(4x - 1)(x - 1) = 0$$
$$x = \frac{1}{4} \text{ or } x = 1$$

Both solutions check, so the solution set is $\left\{\frac{1}{4}, 1\right\}.$

Let's Review the Absolute Value Function

- The symbol for the absolute value of x is $|x|$
- The absolute value function is a piecewise-defined function.
- The output from the absolute value function is never negative.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$|-5| = 5$

$|-2.3| = 2$

$|0| = 0$

$|5| = 5$

$|2.3| = 2.3$

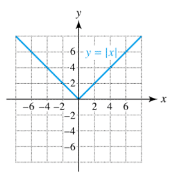


FIGURE 2.63 The Absolute Value Function

How to Solve an Absolute Value Equation?

Let k be a positive number. Then

$$|ax + b| = k$$

is equivalent to

$$ax + b = \pm k$$

Example: Solve $|1 - 2x| = 3$

$$1 - 2x = 3 \text{ or } 1 - 2x = -3$$
$$-2x = 3 - 1 \text{ or } -2x = -3 - 1$$
$$-2x = 2 \text{ or } -2x = -4$$
$$x = -1 \text{ or } x = 2$$

How to Solve Absolute Value Inequalities?

ABSOLUTE VALUE INEQUALITIES

Let the solutions to $|ax + b| = k$ be s_1 and s_2, where $s_1 < s_2$ and $k > 0$.

1. $|ax + b| < k$ is equivalent to $s_1 < x < s_2$.
2. $|ax + b| > k$ is equivalent to $x < s_1$ or $x > s_2$.

Similar statements can be made for inequalities involving $\leq$ or $\geq$.

Example: Solve $|1 - 2x| > 3$

From the previous example the solutions of the equation $|1 - 2x| = 3$ are -1 and 2.

- Thus the solutions for the inequality $|1 - 2x| > 3$ are $x < -1$ or $x > 2$. [Hint: See 2. above.]
- In interval notation this is $(-\infty, -1) \cup (2, \infty)$

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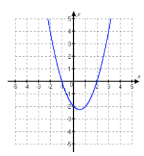
How to Solve a Quadratic Inequality?

A first step in solving a quadratic inequality is to determine the x-values where equality occurs.

These x-values are the *boundary numbers*.

Let's look at a quadratic inequality graphically. The graph of a quadratic function opens either upward or downward. In this case, $a = 1$, parabola opens up, and we have x-intercepts: -1 and 2



$y = x^2 - x - 2$

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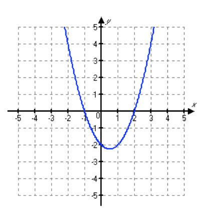
How to Solve a Quadratic Inequality? (Cont.)

Note the parabola lies below the x-axis between the intercepts for the equation $x^2 - x - 2 = 0$



$y = x^2 - x - 2$

Solutions to $x^2 - x - 2 < 0$, is the solution set $\{x|-1 < x < 2\}$ or $(-1, 2)$ in interval notation.

Solutions to $x^2 - x - 2 > 0$ include x-values either left of $x = -1$ or right of $x = 2$, where the parabola is above the x-axis, and thus $\{x|x < -1$ or $x > 2\}$.

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Example

Solve the inequality. Write the solution set for each in interval notation.

a) $3x^2 + x - 4 = 0$ b) $3x^2 + x - 4 < 0$

c) $3x^2 + x - 4 > 0$

Solution

a) Factoring $(3x + 4)(x - 1) = 0$

$x = -4/3 \quad x = 1$

The solutions are – 4/3 and 1.

Example (Cont.)

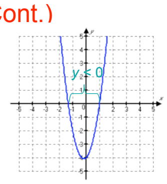
b) $3x^2 + x - 4 < 0$

Parabola opening upward.

x-intercepts are $-4/3$ and 1

Below the x-axis ($y < 0$)

Solution set: $(-4/3, 1)$

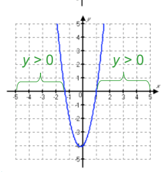


c) $3x^2 + x - 4 > 0$

Above the x axis ($y > 0$)

Solution set:

$(-\infty, -4/3) \cup (1, \infty)$



Solving Quadratic Inequalities in 4 Steps

SOLVING QUADRATIC INEQUALITIES

STEP 1: If necessary, write the inequality as $ax^2 + bx + c < 0$, where $<$ may be replaced by $>$, $\leq$, or $\geq$

STEP 2: Solve the equation $ax^2 + bx + c = 0$ either graphically or symbolically. The solutions are called boundary numbers.

STEP 3: Use the boundary numbers to separate the number line into disjoint intervals. Note that on each interval, $y = ax^2 + bx + c$ is either always positive or always negative.

STEP 4: To solve the inequality, use either a graph or a table. For example, the solution set to $ax^2 + bx + c < 0$ corresponds to intervals where the graph of $y = ax^2 + bx + c$ is below the x-axis or to intervals where test values result in negative values.

What have we learned?

We have learned to:

1. Use properties of rational exponents (rational powers).
2. Understand radical notation.
3. Solve equations involving rational powers.
4. Solve radical equations.
5. Solve absolute value equations
6. Solve inequalities involving absolute values.
7. Solve quadratic inequalities.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition