MAC 1105

Module 8
Exponential and Logarithmic Functions I

Learning Objectives

Upon completing this module, you should be able to:

1. Distinguish between linear and exponential growth.
2. Model data with exponential functions.
3. Calculate compound interest.
4. Use the natural exponential function in applications.
5. Evaluate the common logarithmic function.
6. Evaluate the natural logarithmic function.
7. Solve basic exponential and logarithmic equations.

Exponential and Logarithmic Functions I

There are two major topics in this module:

- Exponential Functions
- Logarithmic Functions

A Quick Review on Function

y = $f(x)$ means that given an input x, there is just one corresponding output y.

Graphically, this means that the graph passes the **vertical line test**.

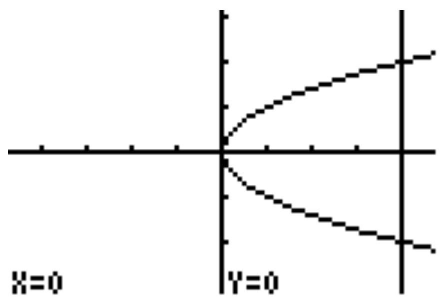
Numerically, this means that in a table of values for $y = f(x)$ there are **no *x*-values repeated**.

Example

Given $y^2 = x$, is $y = f(x)$? That is, is y a function of x?

No, because if $x = 4$, y could be 2 or –2.

Note that the graph fails the vertical line test.



x	*y*
4	**– 2**
1	– 1
0	0
1	1
4	**2**

Note that there is a value of ***x*** in the table for which there are two different values of ***y*** (that is, ***x***-values are repeated.)

What is One-to-One?

- Given a function $y = f(x)$, f is one-to-one means that given an output y there was just one input x which produced that output.
- Graphically, this means that the graph passes the **horizontal line test**. Every horizontal line intersects the graph at most once.
- Numerically, this means the there are **no *y*-values repeated** in a table of values.

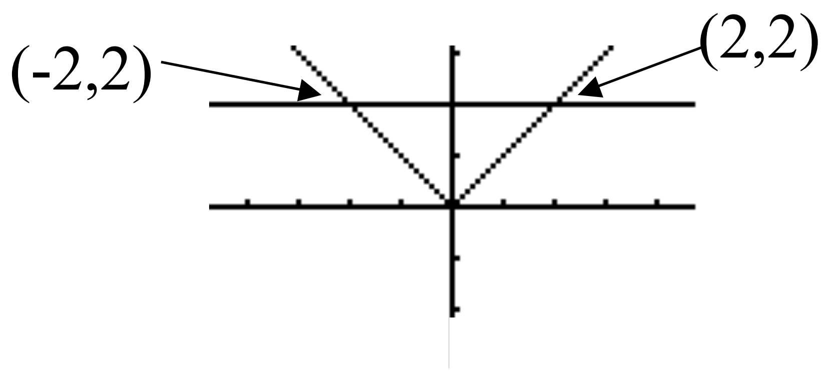
Example

Given $y = f(x) = |x|$, is f one-to-one?

No, because if $y = 2$, x could be 2 or – 2.

Note that the graph fails the horizontal line test.

x	***y***
– 2	2
– 1	1
0	0
1	1
2	2



Note that there is a value of ***y*** in the table for which there are two different values of ***x*** (that is, *y*-values are repeated.)

What is the Definition of a One-to-One Function?

A function f is a one-to-one function if, for elements c and d in the domain of f,

$$c \neq d \text{ implies } f(c) \neq f(d)$$

Example: Given $y = f(x) = |x|$, f is not one-to-one because $-2 \neq 2$ yet $|-2| = |2|$

What is an Inverse Function?

- f^{-1} is a symbol for the inverse of the function f, not to be confused with the reciprocal.
- If $f^{-1}(x)$ does <u>NOT</u> mean $1/f(x)$, what does it mean?
  - **$y = f^{-1}(x)$ means that $x = f(y)$**
  - Note that $y = f^{-1}(x)$ is pronounced "y equals f inverse of x."

Example of an Inverse Function

Let F be Fahrenheit temperature and let C be Centigrade temperature.

$F = f(C) = (9/5)C + 32$

$C = f^{-1}(F) = ?????$

The function f multiplies an input C by 9/5 and adds 32.

To undo multiplying by 9/5 and adding 32, one should subtract 32 and divide by 9/5

So $C = f^{-1}(F) = (F - 32)/(9/5)$

$C = f^{-1}(F) = (5/9)(F - 32)$

Example of an Inverse Function (Cont.)

$F = f(C) = (9/5)C + 32$

$C = f^{-1}(F) = (5/9)(F - 32)$

Evaluate $f(0)$ and interpret.

$f(0) = (9/5)(0) + 32 = 32$

When the Centigrade temperature is 0, the Fahrenheit temperature is 32.

Evaluate $f^{-1}(32)$ and interpret.

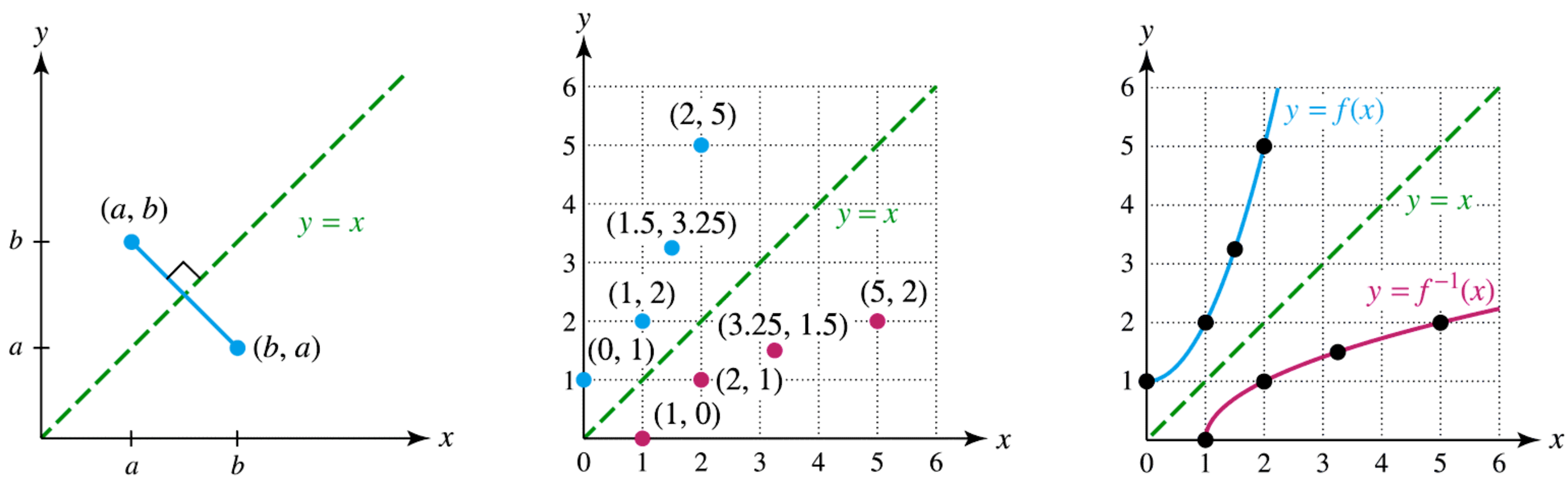
$f^{-1}(32) = (5/9)(32 - 32) = 0$

When the Fahrenheit temperature is 32, the Centigrade temperature is 0.

Note that $f(0) = \mathbf{32}$ and $f^{-1}(\mathbf{32}) = 0$

Graph of Functions and Their Inverses

The graph of f^{-1} is a reflection of the graph of f across the line $y = x$



Note that the domain of f equals the range of f^{-1} and the range of f equals the domain of f^{-1} .

How to Find Inverse Function Symbolically?

Check that f is a one-to-one function. If not, f^{-1} does not exist.

Solve the equation $y = f(x)$ for x, resulting in the equation $x = f^{-1}(y)$

Interchange x and y to obtain $y = f^{-1}(x)$

Example.

$f(x) = 3x + 2$

$y = 3x + 2$

Solving for x gives: $3x = y – 2$

$x = (y – 2)/3$

Interchanging x and y gives: $y = (x – 2)/3$

So $f^{-1}(x) = (x – 2)/3$

How to Evaluate Inverse Function Numerically?

x	$f(x)$
1	–5
2	–3
3	0
4	3
5	5

The function is one-to-one, so f^{-1} exists.

$f^{-1}(-5) = 1$

$f^{-1}(-3) = 2$

$f^{-1}(0) = 3$

$f^{-1}(3) = 4$

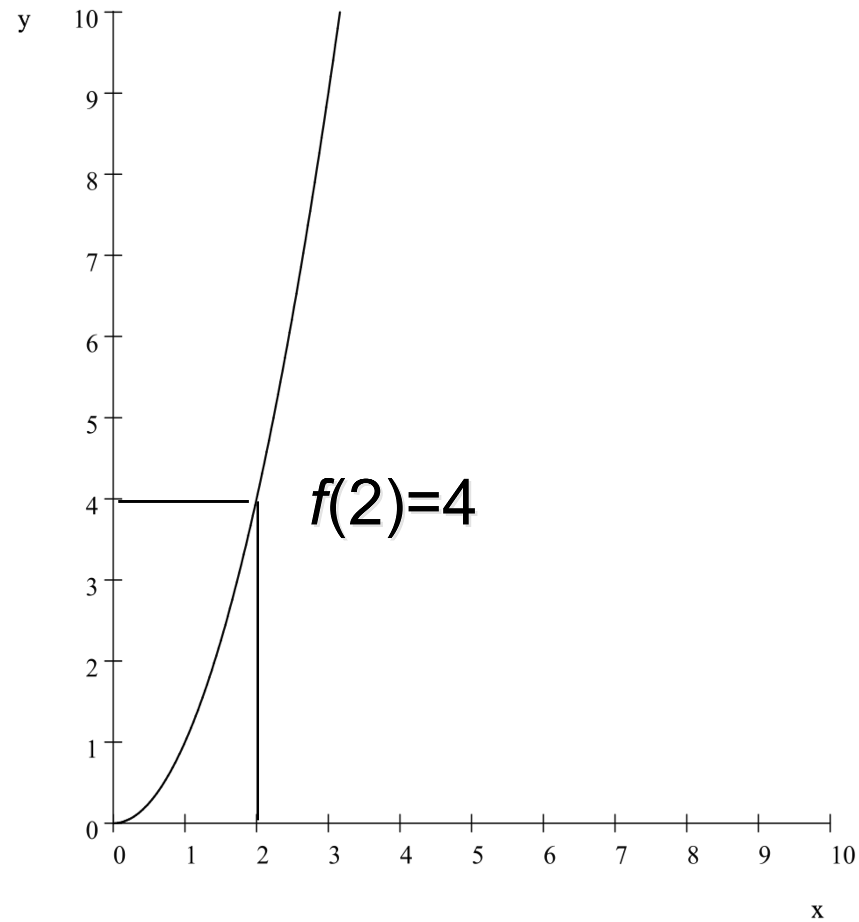
$f^{-1}(5) = 5$

How to Evaluate Inverse Function Graphically?

The graph of *f* below passes the horizontal line test so *f* is one-to-one.

Evaluate $f^{-1}(4)$.

Since the point (2,4) is on the graph of *f*, the point (4,2) will be on the graph of f^{-1} and thus $f^{-1}(4) = 2$.



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What is the Formal Definition of Inverse Functions?

Let f be a one-to-one function. Then f^{-1} is the inverse function of f, if

- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for every x in the domain of f
- $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

Exponential Functions and Models

We will start with population growth.

Population Growth

Suppose a population is 10,000 in January 2004.
Suppose the population increases by…

500 people per year

What is the population in Jan 2005?

10,000 + 500 = 10,500

What is the population in Jan 2006?

10,500 + 500 = 11,000

5% per year

What is the population in Jan 2005?

10,000 + .05(10,000) = 10,000 + 500 = 10,500

What is the population in Jan 2006?

10,500 + .05(10,500) = 10,500 + 525 = 11,025

Population Growth (Cont.)

Suppose a population is 10,000 in Jan 2004. Suppose the population increases by 500 per year. What is the population in ….

Jan 2005?

10,000 + 500 = 10,500

Jan 2006?

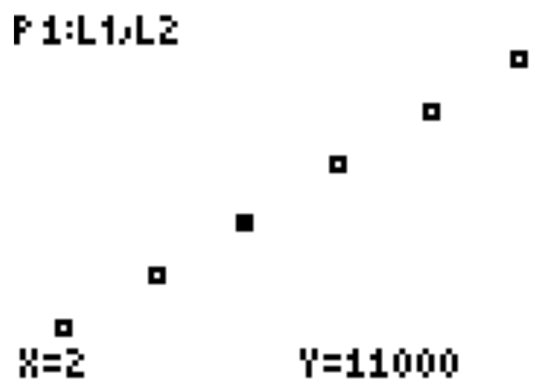
10,000 + 2(500) = 11,000

Jan 2007?

10,000 + 3(500) = 11,500

Jan 2008?

10,000 + 4(500) = 12,000



Population Growth (Cont.)

Suppose a population is 10,000 in Jan 2004 and increases by 500 per year.

Let t be the number of years after 2004. Let $P(t)$ be the population in year t. What is the symbolic representation for $P(t)$? We know…

Population in 2004 = $P(0) = 10{,}000 + 0(500)$

Population in 2005 = $P(1) = 10{,}000 + 1(500)$

Population in 2006 = $P(2) = 10{,}000 + 2(500)$

Population in 2007 = $P(3) = 10{,}000 + 3(500)$

Population t years after 2004 =
$P(t) = 10{,}000 + t(500)$

Population Growth (Cont.)

Population is 10,000 in 2004; increases by 500 per year
$P(t) = 10,000 + t(500)$

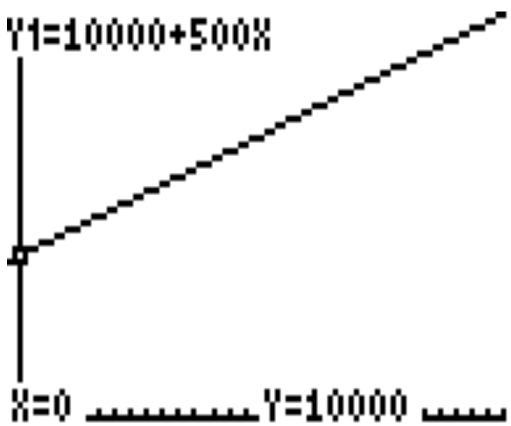
P is a linear function of *t*.

What is the slope?

500 people/year

What is the *y*-intercept?

number of people at time 0 (the year 2004) = 10,000



When *P* increases by a constant number of people per year, *P* is a linear function of *t*.

Population Growth (Cont.)

Suppose a population is 10,000 in Jan 2004 and increases by 5% per year.

Jan 2005?

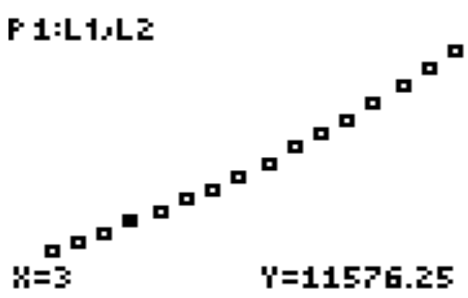
10,000 + .05(10,000) = 10,000 + 500 = 10,500

Jan 2006?

10,500 + .05(10,500) = 10,500 + 525 = 11,025

Jan 2007?

11,025 + .05(11,025) = 11,025 + 551.25 = 11,576.25



Population Growth (Cont.)

Suppose a population is 10,000 in Jan 2004 and increases by 5% per year.

Let t be the number of years after 2004. Let $P(t)$ be the population in year t. What is the symbolic representation for $P(t)$? We know…

Population in 2004 = $P(0) = 10{,}000$

Population in 2005 = $P(1) = 10{,}000 + .05\,(10{,}000) = 1.05(10{,}000) = 1.05^1(10{,}000) = 10{,}500$

Population in 2006 = $P(2) = 10{,}500 + .05\,(10{,}500) = 1.05\,(10{,}500) = 1.05\,(1.05)(10{,}000) = 1.05^2(10{,}000) = 11{,}025$

Population t years after 2004 = $P(t) = 10{,}000(1.05)^t$

Population Growth (Cont.)

Population is 10,000 in 2004; increases by 5% per year

$P(t) = 10{,}000\,(1.05)^t$

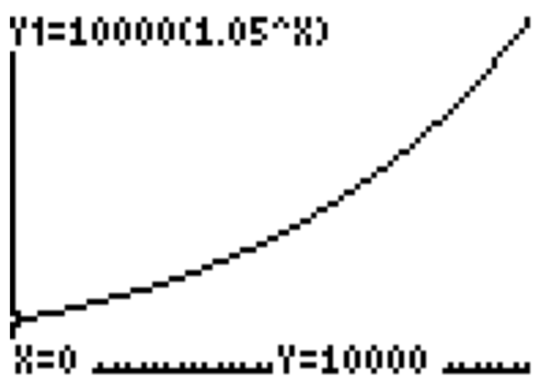
P is an EXPONENTIAL function of t. More specifically, an exponential growth function.

What is the base of the exponential function?

1.05

What is the y-intercept?

number of people at time 0 (the year 2004) = 10,000



When P increases by a constant percentage per year, P is an exponential function of t.

The Main Difference Between a Linear Growth and an Exponential Growth

- A Linear Function adds a fixed amount to the previous value of *y* for each unit increase in *x*
- For example, in $f(x) = 10{,}000 + 500x$ 500 is added to *y* for each increase of 1 in x.

- An Exponential Function multiplies a fixed amount to the previous value of *y* for each unit increase in *x*.
- For example, in $f(x) = 10{,}000\ (1.05)^x$ *y* is multiplied by 1.05 for each increase of 1 in *x*.

The Definition of an Exponential Function

A function represented by

$f(x) = Ca^x$, $a > 0$, a is not 1, and $C > 0$ is an exponential function with base a and coefficient C.

If $a > 1$, then f is an exponential growth function

If $0 < a < 1$, then f is an exponential decay function

What is the Common Mistake?

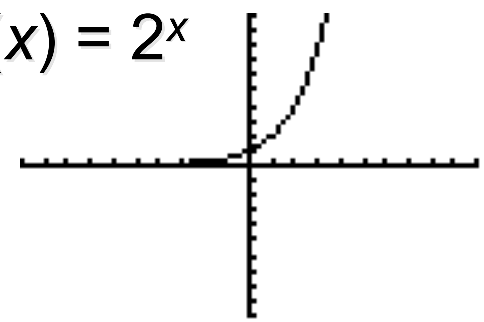
Don't confuse $f(x) = 2^x$ with $f(x) = x^2$

$f(x) = 2^x$ is an exponential function.

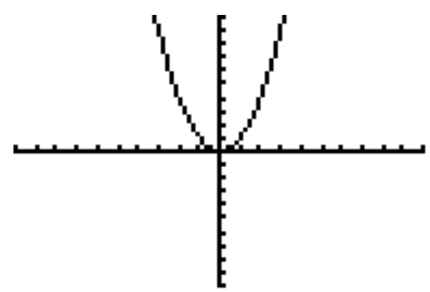
$f(x) = x^2$ is a polynomial function, specifically a quadratic function.

The functions and consequently their graphs are very different.

$f(x) = 2^x$



$f(x) = x^2$



Exponential Growth vs. Decay

- Example of exponential growth function

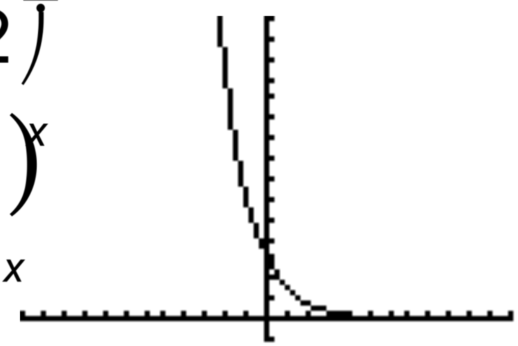
  $f(x) = 3 \cdot 2^x$

- Example of exponential decay function

  $$f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$$

  $$f(x) = 3\left(2^{-1}\right)^x$$

  $$f(x) = 3 \cdot 2^{-x}$$



Recall, in the exponential function $f(x) = Ca^x$

If $a > 1$, then f is an exponential growth function

If $0 < a < 1$, then f is an exponential decay function

Properties of an Exponential Growth Function

Example

$f(x) = 3 \cdot 2^x$

Properties of an exponential growth function:

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- *f* increases on $(-\infty, \infty)$
- The negative *x*-axis is a horizontal asymptote.
- *y*-intercept is (0,3).

Properties of an Exponential Decay Function

Example

$$f(x) = 3 \times \left(\frac{1}{2}\right)^x$$

$$f(x) = 3\left(2^{-1}\right)^x$$

$$f(x) = 3 \times 2^{-x}$$

Properties of an exponential decay function:

- Domain: (-∞, ∞)
- Range: (0, ∞)
- *f* decreases on (-∞, ∞)
- The positive *x*-axis is a horizontal asymptote.
- *y*-intercept is (0,3).

Example of an Exponential Decay: Carbon-14 Dating

The time it takes for half of the atoms to decay into a different element is called the half-life of an element undergoing radioactive decay.

The half-life of carbon-14 is 5700 years.

Suppose C grams of carbon-14 are present at $t = 0$. Then after 5700 years there will be $C/2$ grams present.

Example of an Exponential Decay: Carbon-14 Dating (Cont.)

Let t be the number of years.

Let $A = f(t)$ be the amount of carbon-14 present at time t.

Let C be the amount of carbon-14 present at $t = 0$.

Then $f(0) = C$ and $f(5700) = C/2$.

Thus two points of f are $(0, C)$ and $(5700, C/2)$

Using the point $(5700, C/2)$ and substituting 5700 for t and $C/2$ for A in $A = f(t) = Ca^t$ *yields:*

$$C/2 = C\,a^{5700}$$

Dividing both sides by C yields: $1/2 = a^{5700}$

Example of an Exponential Decay: Carbon-14 Dating (Cont.)

$$\frac{1}{2} = a^{5700}$$

Raising both sides to the 1/5700 power gives

$$\left(\frac{1}{2}\right)^{\frac{1}{5700}} = a$$

So $A = f(t) = Ca^t$ becomes

$$A = f(t) = C\left[\left(\frac{1}{2}\right)^{\frac{1}{5700}}\right]^t$$

$$A = f(t) = C\left[\left(\frac{1}{2}\right)^{\frac{t}{5700}}\right]$$

Half-life

Radioactive Decay
(An Exponential Decay Model)

If a radioactive sample containing *C* units has a half-life of *k* years, then the amount *A* remaining after *x* years is given by

$$A(x) = C\left(\frac{1}{2}\right)^{\frac{x}{k}}$$

Example of Radioactive Decay

Radioactive strontium-90 has a half-life of about 28 years and sometimes contaminates the soil. After 50 years, what percentage of a sample of radioactive strontium would remain?

$$A(x) = C\left(\frac{1}{2}\right)^{\frac{x}{k}}$$

Note calculator keystrokes:

```
.5^(50/28)
      .2900323465
```

$$A(50) = C\left(\frac{1}{2}\right)^{\frac{50}{28}} \approx C(.2900323465)$$

Since C is present initially and after 50 years .29C remains, then 29% remains.

Example of an Exponential Growth: Compound Interest

Suppose $10,000 is deposited into an account which pays 5% interest compounded annually. Then the amount *A* in the account after *t* years is:

$$A(t) = 10{,}000\,(1.05)^t$$

Note the similarity with: Suppose a population is 10,000 in 2004 and increases by 5% per year. Then the population *P*, *t* years after 2004 is:

$$P(t) = 10{,}000\,(1.05)^t$$

What is the Natural Exponential Function?

The function *f*, represented by

$$f(x) = e^x$$

is the natural exponential function where

$e \approx 2.718281828$

Example of Using Natural Exponential Function

Suppose $100 is invested in an account with an interest rate of 8% compounded continuously. How much money will there be in the account after 15 years?

In this case, P = $100, r = 8/100 = 0.08 and t = 15 years. Thus,

$$A = Pe^{rt}$$

$$A = \$100\, e^{.08(15)}$$

$$A = \$332.01$$

Logarithmic Functions and Models

What is the Definition of a Common Logarithmic Function?

The common logarithm of a positive number x, denoted $\log(x)$, is defined by

$$\log(x) = k \text{ if and only if } x = 10^k$$

where k is a real number.

The function given by $f(x) = \log(x)$ is called the common logarithmic function.

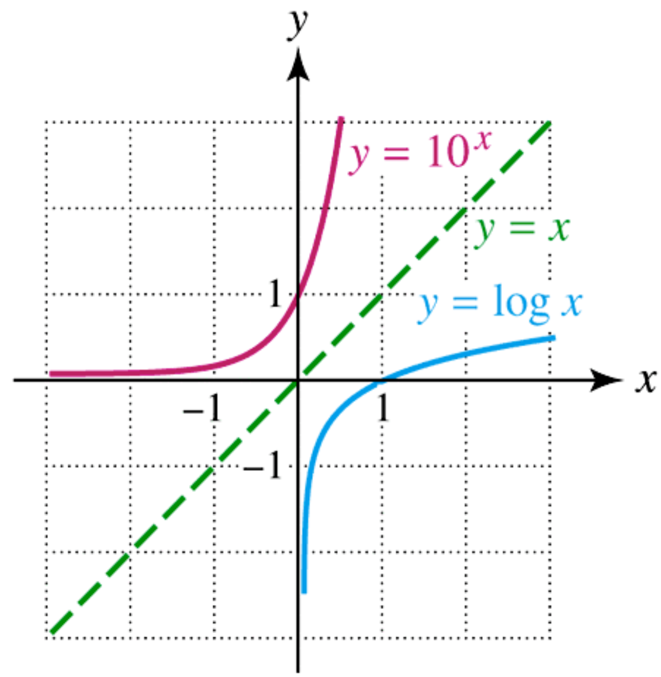
Note that the input x must be positive.

Let's Evaluate Some Common Logarithms

log (10)	**1** because $10^{1} = 10$
log (100)	**2** because $10^{2} = 100$
log (1000)	**3** because $10^{3} = 1000$
log (10000)	**4** because $10^{4} = 10000$
log (1/10)	**–1** because $10^{-1} = 1/10$
log (1/100)	**–2** because $10^{-2} = 1/100$
log (1/1000)	**–3** because $10^{-3} = 1/1000$
log (1)	**0** because $10^{0} = 1$

Graph of a Logarithmic Function

x	$f(x)$
0.01	-2
0.1	-1
1	0
10	1
100	2



Note that the graph of $y = \log(x)$ is the graph of $y = 10^x$ reflected through the line $y = x$. This suggests that these are inverse functions.

What is the Inverse Function of a Common Logarithmic Function?

Note that the graph of $f(x) = \log(x)$ passes the horizontal line test so it is a one-to-one function and has an inverse function.

Find the inverse of $y = \log(x)$

Using the definition of common logarithm to solve for x gives

$$x = 10^y$$

Interchanging x and y gives

$$y = 10^x$$

Thus, the inverse of $y = \log(x)$ is $y = 10^x$

What is the Inverse Properties of the Common Logarithmic Function?

Recall that $f^{-1}(x) = 10^x$ given $f(x) = \log(x)$

Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}

$\log(10^x) = x$ for all real numbers x.

Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f

$10^{\log x} = x$ for any positive number x

What is the Definition of a Logarithmic Function with base a?

- The logarithm with base a of a positive number x, denoted by $\log_a(x)$ is defined by

  $\log_a(x) = k$ if and only if $x = a^k$

  where $a > 0$, $a \neq 1$, and k is a real number.

- The function given by $f(x) = \log_a(x)$ is called the logarithmic function with base a.

What is the Natural Logarithmic Function?

- Logarithmic Functions with Base 10 are called "common logs."
  - $\log(x)$ means $\log_{10}(x)$ - *The Common Logarithmic Function*
- Logarithmic Functions with Base e are called "natural logs."
  - $\ln(x)$ means $\log_e(x)$ - *The Natural Logarithmic Function*

Let's Evaluate Some Natural Logarithms

- ln (e) — ln (e) = $\log_e(e)$ = 1 since $e^1 = e$
- ln (e^2) — $\ln(e^2) = \log_e(e^2) = 2$ since 2 is the exponent that goes on e to produce e^2.
- ln (1) — ln (1) = $\log_e 1 = 0$ since $e^0 = 1$
- $\ln\sqrt{e}$ — 1/2 since 1/2 is the exponent that goes on e to produce $e^{1/2}$

What is the Inverse of a Logarithmic Function with base a?

Note that the graph of $f(x) = \log_a(x)$ passes the horizontal line test so it is a one-to-one function and has an inverse function.

Find the inverse of $y = \log_a(x)$

Using the definition of common logarithm to solve for x gives

$$x = a^y$$

Interchanging x and y gives

$$y = a^x$$

Thus, the inverse of $y = \log_a(x)$ is $y = a^x$

What is the Inverse Properties of a Logarithmic Function with base a?

Recall that $f^{-1}(x) = a^x$ given $f(x) = \log_a(x)$

Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}

$\log_a(a^x) = x$ for all real numbers x.

Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f

$a^{\log_a x} = x$ for any positive number x

Let's Try to Solve Some Equations

Solve the equation $4^x = 1/64$

Take the log of both sides to the base 4

$\log_4 (4^x) = \log_4(1/64)$

Using the inverse property $\log_a (a^x) = x$, this simplifies to

$x = \log_4(1/64)$

Since 1/64 can be rewritten as 4^{-3}

$x = \log_4(4^{-3})$

Using the inverse property $\log_a (a^x) = x$, this simplifies to

$x = -3$

Let's Try to Solve Some Equations

Solve the equation $e^x = 15$

Take the log of both sides to the base e

$\ln(e^x) = \ln(15)$

Using the inverse property $\log_a(a^x) = x$ this simplifies to

$x = \ln(15)$

Using the calculator to estimate ln (15)

$x \approx 2.71$

Let's Try to Solve Some Equations (Cont.)

Solve the equation ln $(x) = 1.5$

Exponentiate both sides using base e

$$e^{\ln x} = e^{1.5}$$

Using the inverse property $a^{\log_a x} = x$ this simplifies to

$$x = e^{1.5}$$

Using the calculator to estimate $e^{1.5}$

$$x \approx 4.48$$

What have we learned?

We have learned to:

1. Distinguish between linear and exponential growth.
2. Model data with exponential functions.
3. Calculate compound interest.
4. Use the natural exponential function in applications.
5. Evaluate the common logarithmic function.
6. Evaluate the natural logarithmic function.
7. Solve basic exponential and logarithmic equations.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition