MAC 1105

Module 9
Exponential and
Logarithmic Functions II

Learning Objective

Upon completing this module, you should be able to:

1. Learn and apply the basic properties of logarithms.
2. Use the change of base formula to compute logarithms.
3. Solve an exponential equation by writing it in logarithmic form and/or using properties of logarithms.
4. Solve logarithmic equations.
5. Apply exponential and logarithmic functions in real world situations.

Exponential and Logarithmic Functions II

There are two major sections in this module:

- Properties of Logarithms
- Exponential Functions and Investing

Property 1

- $\log_a(1) = 0$ and $\log_a(a) = 1$
  - $a^0 = 1$ and $a^1 = a$
- Note that this property is a direct result of the inverse property $\log_a(a^x) = x$
- Example: $\log(1) = 0$ and $\ln(e) = 1$

Property 2

- $\log_a(m) + \log_a(n) = log_a(mn)$

- The sum of logs is the log of the product.

- Example: Let $a = 2$, $m = 4$ and $n = 8$
- $\log_a(m) + \log_a(n) = \log_2(4) + \log_2(8) = 2 + 3$
- $log_a(mn) = log_2(4 \cdot 8) = log_2(32) = 5$

Property 3

- $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$

- The difference of logs is the log of the quotient.
- Example: Let $a = 2$, $m = 4$ and $n = 8$

$$\log_a m - \log_a n = \log_2 4 - \log_2 8 = 2 - 3 = -1$$

$$\log_a\left(\frac{m}{n}\right) = \log_2\left(\frac{4}{8}\right) = \log_2\left(\frac{1}{2}\right) = -1$$

Property 4

- $\log_a\left(m^r\right) = r\log_a m$

- Example: Let $a = 2$, $m = 4$ and $r = 3$

$$\log_a\left(m^r\right) = \log_2\left(4^3\right) = \log_2(64) = 6$$

$$r\log_a m = 3\log_2 4 = 3(2) = 6$$

Example

- Expand the expression. Write without exponents.

$$\log\left(\frac{3x^6}{2y^7}\right)$$

$$\log\left(\frac{3x^6}{2y^7}\right) = \log\left(3x^6\right) - \log\left(2y^7\right)$$

$$\log 3 + \log\left(x^6\right) - \left(\log 2 + \log\left(y^7\right)\right)$$

$$\log 3 + 6\log x - \log 2 - 7\log y$$

Example

- Write as the logarithm of a single expression

$$2\ln x - 4\ln y + \frac{1}{2}\ln z$$

$$2\ln x - 4\ln y + \frac{1}{2}\ln z = \ln\left(x^2\right) - \ln\left(y^4\right) + \ln\left(z^{\frac{1}{2}}\right)$$

$$= \ln\left(\frac{x^2}{y^4}\right) + \ln\left(z^{\frac{1}{2}}\right) = \ln\left(\frac{x^2}{y^4} \times z^{\frac{1}{2}}\right) = \ln\left(\frac{x^2 \times z^{\frac{1}{2}}}{y^4}\right)$$

$$= \ln\left(\frac{x^2 \times \sqrt{z}}{y^4}\right)$$

Change of Base Formula

Let x, $a \neq 1$, and $b \neq 1$ be positive real numbers. Then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example of Using the Change of Base Formula

- Use the change of base formula to evaluate $\log_3 8$

$$\log_3 8 = \frac{\log_{10} 8}{\log_{10} 3} = \frac{\log 8}{\log 3} \approx 1.893$$

Solve $3(1.2)^x + 2 = 15$ for x symbolically by Writing it in Logarithmic Form

$$3(1.2)^x = 13$$

$$1.2^x = \frac{13}{3}$$

Divide each side by 3

$$\log 1.2^x = \log\left(\frac{13}{3}\right)$$

Take common logarithm of each side

(Could use natural logarithm)

$$x \log 1.2 = \log\left(\frac{13}{3}\right)$$

Use Property 4: $\log(m^r) = r \log(m)$

$$x = \frac{\log\left(\frac{13}{3}\right)}{\log 1.2}$$

Divide each side by log (1.2)

Approximate using calculator

$$x \approx 8.04$$

Solve $e^{x+2} = 5^{2x}$ for x Symbolically by Writing it in Logarithmic Form

$$e^{x+2} = 5^{2x}$$

$$\ln\left(e^{x+2}\right) = \ln\left(5^{2x}\right)$$
Take natural logarithm of each side

$$(x+2)\ln e = 2x\ln 5$$
Use Property 4: ln (m^r) = r ln (m)

$$x + 2 = 2x\ln 5$$
ln (e) = 1

$$x - 2x\ln 5 = -2$$
Subtract $2x$ ln(5) and 2 from each side

$$x(1 - 2\ln 5) = -2$$
Factor x from left-hand side

$$x = \frac{-2}{1 - 2\ln 5}$$
Divide each side by 1 – 2 ln (5)

$$x \approx .901$$
Approximate using calculator

Solving a Logarithmic Equation Symbolically

- In developing countries there is a relationship between the amount of land a person owns and the average daily calories consumed. This relationship is modeled by the formula $C(x) = 280 \ln(x+1) + 1925$ where x is the amount of land owned in acres and

  Source: D. Gregg: *The World Food Problem*

- Determine the number of acres owned by someone whose average intake is 2400 calories per day.
- Must solve for *x* in the equation

$$280 \ln(x+1) + 1925 = 2400$$

Solving a Logarithmic Equation Symbolically (Cont.)

$$280\ln(x+1)+1925=2400$$

$$280\ln(x+1)=2400-1925$$ Subtract 1925 from each side

$$280\ln(x+1)=475$$

$$\ln(x+1)=\frac{475}{280}$$ Divide each side by 280

$$e^{\ln(x+1)}=e^{\frac{475}{280}}$$ Exponentiate each side base *e*

Inverse property $e^{\ln k}=k$

$$x+1=e^{\frac{475}{280}}$$ Subtract 1 from each side

$$x=e^{\frac{475}{280}}-1$$ Approximate using calculator

$$x\approx 4.45$$

Quick Review of
Exponential Growth/Decay Models

Example of an Exponential Decay: Carbon-14 Dating

The time it takes for half of the atoms to decay into a different element is called the half-life of an element undergoing radioactive decay.

The half-life of carbon-14 is 5700 years.

Suppose C grams of carbon-14 are present at $t = 0$. Then after 5700 years there will be $C/2$ grams present.

Example of an Exponential Decay: Carbon-14 Dating (Cont.)

Let t be the number of years.

Let $A = f(t)$ be the amount of carbon-14 present at time t.

Let C be the amount of carbon-14 present at $t = 0$.

Then $f(0) = C$ and $f(5700) = C/2$.

Thus two points of f are $(0, C)$ and $(5700, C/2)$

Using the point $(5700, C/2)$ and substituting 5700 for t and $C/2$ for A in $A = f(t) = Ca^t$ *yields:*

$$C/2 = C\,a^{5700}$$

Dividing both sides by C yields: $1/2 = a^{5700}$

Example of an Exponential Decay: Carbon-14 Dating (Cont.)

$$\frac{1}{2} = a^{5700}$$

Raising both sides to the 1/5700 power gives

$$\left(\frac{1}{2}\right)^{\frac{1}{5700}} = a$$

So $A = f(t) = Ca^t$ becomes

$$A = f(t) = C\left[\left(\frac{1}{2}\right)^{\frac{1}{5700}}\right]^t$$

$$A = f(t) = C\left[\left(\frac{1}{2}\right)^{\frac{t}{5700}}\right]$$

Half-life

Radioactive Decay (An Exponential Decay Model)

If a radioactive sample containing *C* units has a half-life of *k* years, then the amount *A* remaining after *x* years is given by

$$A(x) = C\left(\frac{1}{2}\right)^{\frac{x}{k}}$$

Example of Radioactive Decay

Radioactive strontium-90 has a half-life of about 28 years and sometimes contaminates the soil. After 50 years, what percentage of a sample of radioactive strontium would remain?

$$A(x) = C\left(\frac{1}{2}\right)^{\frac{x}{k}}$$

Note calculator keystrokes:

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.5^(50/28)
        .2900323465
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$$A(50) = C\left(\frac{1}{2}\right)^{\frac{50}{28}} \approx C(.2900323465)$$

Since C is present initially and after 50 years .29C remains, then 29% remains.

Example of an Exponential Growth: Compound Interest

Suppose $10,000 is deposited into an account which pays 5% interest compounded annually. Then the amount *A* in the account after *t* years is:

$$A(t) = 10,000\ (1.05)^t$$

What is the Compound Interest Formula?

- If P dollars is deposited in an account paying an annual rate of interest r, compounded (paid) n times per year, then after t years the account will contain A dollars, where

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- **Frequencies of Compounding (Adding Interest)**
- annually (1 time per year)
- semiannually (2 times per year)
- quarterly (4 times per year)
- monthly (12 times per year)
- daily (365 times per year)

Example: Compounded Periodically

Suppose $1000 is deposited into an account yielding 5% interest compounded at the following frequencies. How much money after *t* years? $A = P\left(1+\frac{r}{n}\right)^{nt}$

- Annually $\quad A = 1000\left(1+\frac{.05}{1}\right)^{1t} = 1000(1.05)^{t}$
- Semiannually $\quad A = 1000\left(1+\frac{.05}{2}\right)^{2t} = 1000(1.025)^{2t}$
- Quarterly $\quad A = 1000\left(1+\frac{.05}{4}\right)^{4t} = 1000(1.0125)^{4t}$
- Monthly $\quad A = 1000\left(1+\frac{.05}{12}\right)^{12t} = 1000(1.0041\overline{6})^{12t}$

Example: Compounded Continuously

Suppose \$100 is invested in an account with an interest rate of 8% compounded continuously. How much money will there be in the account after 15 years?

In this case, P = \$100, r = 8/100 = 0.08 and t = 15 years. Thus,

$$A = Pe^{rt}$$

$$A = \$100\ e^{.08(15)}$$

$$A = \$332.01$$

Another Example

- How long does it take money to grow from $100 to $200 if invested into an account which compounds quarterly at an annual rate of 5%?
- Must solve for *t* in the following equation

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

Another Example (Cont.)

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

$$2 = (1.0125)^{4t}$$

Divide each side by 100

$$\log 2 = \log(1.0125)^{4t}$$

Take common logarithm of each side

$$\log 2 = 4t \log 1.0125$$

Property 4: $\log(m^r) = r \log(m)$

$$4t \log 1.0125 = \log 2$$

$$t = \frac{\log 2}{4 \log 1.0125}$$

Divide each side by 4log1.0125

$$t \approx 13.95 \text{ years}$$

Approximate using calculator

Another Example (Cont.)

Alternatively, we can take natural logarithm of each side instead of taking the common logarithm of each side.

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

Divide each side by 100

$$2 = (1.0125)^{4t}$$

Take natural logarithm of each side

$$\ln 2 = \ln(1.0125)^{4t}$$

Property 4: ln (m^r) = r ln (m)

$$\ln 2 = 4t \ln 1.0125$$

Divide each side by 4 ln (1.0125)

$$4t \ln 1.0125 = \ln 2$$

$$t = \frac{\ln 2}{4 \ln 1.0125}$$

Approximate using calculator

$$t \approx 13.95 \text{ years}$$

What have we learned?

We have learned to:

1. Learn and apply the basic properties of logarithms.
2. Use the change of base formula to compute logarithms.
3. Solve an exponential equation by writing it in logarithmic form and/or using properties of logarithms.
4. Solve logarithmic equations.
5. Apply exponential and logarithmic functions in real world situations.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition