MAC 2103

Module 11

Inner Product Spaces II

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Learning Objectives

Upon completing this module, you should be able to:

1. Construct an orthonormal set of vectors from an orthogonal set of vectors.
2. Find the coordinate vector with respect to a given orthonormal basis.
3. Construct an orthogonal basis from a nonstandard basis in $\Re^n$ using the Gram-Schmidt process.
4. Find the least squares solution to a linear system A**x** = **b**.
5. Find the orthogonal projection on col(A).
6. Obtain the best approximation.

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General Vector Spaces II

The major topics in this module:

Orthogonal Bases, Gram-Schmidt Process, Least Squares and Best Approximation

How to Construct an Orthonormal Set of Vectors from an Orthogonal Set of Vectors?

We have learned from the previous module that two vectors **u** and **v** in an inner product space V are orthogonal to each other iff <**u**,**v**> = 0.

To obtain an orthonormal set, we will normalize each of the vectors in the orthogonal set.

How to normalize the vectors? This can be done by dividing each of them by their respective norm and making each of them a unit vector.

How to Construct an Orthonormal Set of Vectors from an Orthogonal Set of Vectors? (Cont.)

Example 1: Find the orthonormal set of vectors from the following set of vectors: Let S ={$\mathbf{v}_1$, $\mathbf{v}_2$} where $\mathbf{v}_1$ = (5,0) and $\mathbf{v}_2$ = (0,-3).

Step 1: Verify that the set of vectors are mutually orthogonal with respect to the Euclidean inner product on $\Re^2$.

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1 \cdot \vec{v}_2 = (5)(0) + (0)(-3) = 0 \quad S \text{ is an orthogonal set.}$$

Step 2: Find the norm for both vectors.

$$\|\vec{v}_1\| = \sqrt{5^2 + 0^2} = 5, \text{ and}$$

$$\|\vec{v}_2\| = \sqrt{0^2 + (-3)^2} = 3.$$

How to Construct an Orthonormal Set of Vectors from an Orthogonal Set of Vectors? (Cont.)

Step 3: Normalize the vectors in the orthogonal set.

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \left(\frac{5}{5}, \frac{0}{5}\right) = (1,0), \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \left(\frac{0}{3}, \frac{-3}{3}\right) = (0,-1)$$

Step 4: Verify that the set S is orthonormal by showing that $\langle \vec{q}_1, \vec{q}_2 \rangle = 0$ and $\|\vec{q}_1\| = \|\vec{q}_2\| = 1$.

$$\langle \vec{q}_1, \vec{q}_2 \rangle = \vec{q}_1 \cdot \vec{q}_2 = (1)(0) + (0)(-1) = 0,$$

$$\|\vec{q}_1\| = \langle \vec{q}_1, \vec{q}_1 \rangle^{\frac{1}{2}} = (\vec{q}_1 \cdot \vec{q}_1)^{\frac{1}{2}} = \sqrt{1^2 + 0^2} = 1, \text{ and}$$

$$\|\vec{q}_2\| = \langle \vec{q}_2, \vec{q}_2 \rangle^{\frac{1}{2}} = (\vec{q}_2 \cdot \vec{q}_2)^{\frac{1}{2}} = \sqrt{0^2 + (-1)^2} = 1.$$

Orthonormal Set, Orthonormal Basis, and Orthogonal Basis

Orthonormal Set: An orthogonal set in which each vector is a unit vector.

Orthonormal basis: A basis consisting of orthonormal vectors in an inner product space. **Example:** The standard basis for $\Re^n$.

Orthogonal basis: A basis consisting of orthogonal vectors in an inner product space.

Note that if S is an orthogonal set, then S is a linearly independent set.

Orthonormal Set, Orthonormal Basis, and Orthogonal Basis (Cont.)

If S ={$\mathbf{v}_1$, $\mathbf{v}_2$, … , $\mathbf{v}_n$} is an orthogonal basis of W, then for any **w** ∈ W,

$$\vec{w} = \sum_{i=1}^{n} \frac{\langle \vec{w}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i = \frac{\langle \vec{w}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{w}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \dots \frac{\langle \vec{w}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle} \vec{v}_n,$$

where $\frac{\langle \vec{w}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}, \frac{\langle \vec{w}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}, \dots, \frac{\langle \vec{w}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle}$

are called the Fourier coefficients.

So the coordinate vector of **w**,

$$\vec{w}_S = (\vec{w})_S = \left(\frac{\langle \vec{w}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}, \frac{\langle \vec{w}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}, \dots, \frac{\langle \vec{w}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle} \right).$$

Orthonormal Set, Orthonormal Basis, and Orthogonal Basis (Cont.)

If S ={$\mathbf{q}_1$, $\mathbf{q}_2$, ... , $\mathbf{q}_n$} is an orthonormal basis of W, then for any **w** ∈ W,

$$\vec{w} = \sum_{i=1}^{n} \frac{\langle \vec{w}, \vec{q}_i \rangle}{\langle \vec{q}_i, \vec{q}_i \rangle} \vec{q}_i = \sum_{i=1}^{n} \langle \vec{w}, \vec{q}_i \rangle \vec{q}_i$$

$$= \langle \vec{w}, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{w}, \vec{q}_2 \rangle \vec{q}_2 + \ldots \langle \vec{w}, \vec{q}_n \rangle \vec{q}_n,$$

$$as\ \langle \vec{q}_i, \vec{q}_i \rangle = \|\vec{q}_i\|^2 = 1\ \ for\ i = 1, 2,\ \ldots, n.$$

where $\langle \vec{w}, \vec{q}_1 \rangle, \langle \vec{w}, \vec{q}_2 \rangle,\ \ldots\ , \langle \vec{w}, \vec{q}_n \rangle$ are called the Fourier coefficients.

So the coordinate vector of **w**,

$$\vec{w}_S = (\vec{w})_S = (\langle \vec{w}, \vec{q}_1 \rangle, \langle \vec{w}, \vec{q}_2 \rangle,\ \ldots\ , \langle \vec{w}, \vec{q}_n \rangle).$$

How to Find the Coordinate Vector with Respect to a Given Orthogonal Basis?

Example 2: Compute the coefficients and determine the coordinate vectors in Example 1 for **u** = (10,3).

From Example 1, we have $\mathbf{v}_1$ = (5,0), $\mathbf{v}_2$ = (0,-3) and $\|\vec{v}_1\| = 5,\ and\ \|\vec{v}_2\| = 3.$

In this case, the coefficients are:

$$\frac{\langle \vec{u}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} = \frac{\vec{u} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} = \frac{(10)(5) + (3)(0)}{5^2} = \frac{50}{25} = 2$$

$$\frac{\langle \vec{u}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} = \frac{\vec{u} \cdot \vec{v}_2}{\|\vec{v}_2\|^2} = \frac{(10)(0) + (3)(-3)}{3^2} = \frac{-9}{9} = -1$$

How to Find the Coordinate Vector with Respect to a Given Orthogonal Basis? (Cont.)

So the coordinate vector of **u**,

$$\vec{u}_S = (\vec{u})_S = \left(\frac{\langle \vec{u}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}, \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \right) = (2, -1).$$

We can see that a nice advantage of working with an orthogonal basis is that the coefficients in any basis representation for a vector are immediately known; they are called Fourier coefficients.

How to Find the Coordinate Vector with Respect to a Given Orthonormal Basis?

Example 3: Find the coordinates of **w** = (2,3) relative to the orthonormal basis for $\Re^2$, B = {$\mathbf{v_1}$, $\mathbf{v_2}$}, where

$$\vec{v}_1 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \ \ and \ \ \vec{v}_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

Since B is orthonormal, we have

$$\langle \vec{w}, \vec{v}_1 \rangle = \vec{w} \cdot \vec{v}_1 = (2,3) \cdot (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = (2)(-\frac{1}{\sqrt{2}}) + 3(-\frac{1}{\sqrt{2}}) = -\frac{5}{\sqrt{2}},$$

$$\langle \vec{w}, \vec{v}_2 \rangle = \vec{w} \cdot \vec{v}_2 = (2,3) \cdot (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = (2)(\frac{1}{\sqrt{2}}) + 3(-\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}},$$

$$and \ \ \vec{w}_B = (\vec{w})_B = (\langle \vec{w}, \vec{v}_1 \rangle, \langle \vec{w}, \vec{v}_2 \rangle) = (-\frac{5}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

The Gram-Schmidt Process

Let S = {$\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_m$} with nonzero $\mathbf{u}_i \in \Re^n$ for i = 1, 2, … , m. S does not have to be a linearly independent set. It might be that A = [$\mathbf{u}_1$ $\mathbf{u}_2$ … $\mathbf{u}_m$] is an n x m matrix and the source of S.

The Gram-Schmidt Algorithm: S = {$\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_m$}

1. Let $\vec{v}_1 = \vec{u}_1$.
2. For *k* = 2, 3, … , m, let

$$\vec{v}_k = \vec{u}_k - \sum_{i=1}^{k-1} \frac{\langle \vec{u}_k, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i .$$

If $\vec{v}_k = \vec{0}$, we discard it since it is linearly dependent.

The Gram-Schmidt Process (Cont.)

3. We then have r ≤ m orthogonal and linearly independent vectors in B = {$\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_r$}.

   span(S) = span(B) = W, a r dimensional subspace of $\Re^n$ and B is an orthogonal basis for W. W = col(A) if A = [$\mathbf{u}_1$ $\mathbf{u}_2$ … $\mathbf{u}_m$].

An orthogonal basis for W is {$\mathbf{q}_1$, $\mathbf{q}_2$, …, $\mathbf{q}_r$} where

$$\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

for i =1, 2, … , r.

How to Construct an Orthonormal Basis from a Nonstandard Basis in $\Re^n$ using the Gram-Schmidt Process?

Let $\Re^n$ be the usual Euclidean inner product space of dimension n. Let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ be a nonstandard basis in $\Re^n$.

Step 1: Use the Gram-Schmidt method to construct an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ from the basis vectors $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$.

Step 2: Normalize the orthogonal basis vectors to obtain the orthonormal basis $\{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\}$.

How to Construct an Orthonormal Basis from a Nonstandard Basis in $\Re^n$ using the Gram-Schmidt Process? (Cont.)

Example 4:

The set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{(-1,1,0),(1,2,1),(3,1,1)\}$ is a nonstandard basis in $\Re^3$.

Step 1:

$$\vec{v}_1 = \vec{u}_1 = (-1,1,0),\ \|\vec{v}_1\|^2 = 2,\ W_1 = span(\{\vec{v}_1\}),\ \dim(W_1) = 1.$$

Projections onto W_1 *have one component.*

$$\vec{v}_2 = \vec{u}_2 - \mathbf{proj}_{W_1}(\vec{u}_2) = (1,2,1) - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (1,2,1) - \frac{1}{2}(-1,1,0) = (\frac{3}{2}, \frac{3}{2}, 1),\ \|\vec{v}_2\|^2 = \frac{11}{2},\ W_2 = span(\{\vec{v}_1, \vec{v}_2\})\ \dim(W_2) = 2.$$

Projections onto W_2 *have two components.*

How to Construct an Orthonormal Basis from a Nonstandard Basis in $\Re^n$ using the Gram-Schmidt Process? (Cont.)

$$\vec{v}_3 = \vec{u}_3 - \mathbf{proj}_{W_2}(\vec{u}_3) = (2,1,1) - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (3,1,1) - \frac{(-2)}{2}(-1,1,0) - \frac{7}{11/2}(\frac{3}{2},\frac{3}{2},1)$$

$$= (3,1,1) + (-1,1,0) - \frac{14}{11}(\frac{3}{2},\frac{3}{2},1)$$

$$= (3,1,1) + (-1,1,0) - (\frac{21}{11},\frac{21}{11},\frac{14}{11}) = (\frac{1}{11},\frac{1}{11},-\frac{3}{11}).$$

$$Thus,\ \vec{v}_1 = (-1,1,0),\ \vec{v}_2 = (\frac{3}{2},\frac{3}{2},1),\ \vec{v}_3 = (\frac{1}{11},\frac{1}{11},-\frac{3}{11}).$$

form an orthogonal basis for $\Re^3$.

How to Construct an Orthonormal Basis from a Nonstandard Basis in $\Re^n$ using the Gram-Schmidt Process? (Cont.)

Step 2: Normalize the orthogonal basis vectors to obtain the orthonormal basis B = {$\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$}.

The norms of these vectors are:

$$\|\vec{v}_1\| = \sqrt{2},\ \|\vec{v}_2\| = \sqrt{\frac{11}{2}},\ and\ \|\vec{v}_3\| = \frac{1}{\sqrt{11}}.$$

So an orthonormal basis is B where

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}(-1,1,0) = (-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0),$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \sqrt{\frac{2}{11}}(\frac{3}{2},\frac{3}{2},1) = (\frac{3\sqrt{2}}{2\sqrt{11}},\frac{3\sqrt{2}}{2\sqrt{11}},\frac{\sqrt{2}}{\sqrt{11}}),\ and$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \sqrt{11}(\frac{1}{11},\frac{1}{11},-\frac{3}{11}) = (\frac{\sqrt{11}}{11},\frac{\sqrt{11}}{11},-\frac{3\sqrt{11}}{11}).$$

How to Find the Least Squares Solution?

The linear system A**x** = **b** always has the associated normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ which is consistent and has one or more solutions. Any solution of this system is a least squares solution of A**x** = **b.** Moreover, the orthogonal projection of **b** on W = col(A) is $A\vec{x} = \mathbf{proj}_W(\vec{b})$, where **x** is a least squares solution.

Example 5: Find the least squares solution of the linear system A**x** = **b** given by

$$A = \begin{bmatrix} 1 & 0 \\ -2 & -1 \\ 1 & 4 \end{bmatrix} \; and \; \vec{b} = \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}.$$

Observe that A has two linearly independent column vectors, so A^TA is invertible, and there is a unique solution to $A^TA\mathbf{x} = A^T\mathbf{b}$, which will be our least squares solution to A**x** = **b.**

How to Find the Least Squares Solution? (Cont.)

We have

$$A^TA = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 17 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$$

So the normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ in this case is

$$\begin{bmatrix} 6 & 6 \\ 6 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$$

How to Find the Least Squares Solution? (Cont.)

By Gauss Elimination, the row-echelon form of

$$\left[A^T A \mid A^T \vec{b}\right] = \begin{bmatrix} 6 & 6 & 10 \\ 6 & 17 & 30 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & -5/33 \\ 0 & 1 & 20/11 \end{bmatrix}.$$

The solution to the normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ in this case is $x_1 = -5/33,\ and\ x_2 = 20/11$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x} = (x_1, x_2) = (-5/33, 20/11)$$

is our unique least square solution to A**x** = **b**.

How to Find the Orthogonal Projection and Obtain the Best Approximation?

$A\vec{x} = \mathbf{proj}_W(\vec{b})$ is the orthogonal projection of **b** on W = col(A). The $\mathbf{proj}_W(\vec{b})$ is the **Best Approximation** to **b** in W since the distance between **b** and $\mathbf{proj}_W(\vec{b})$ is the minimum for all vectors in W,

$$\left\|\vec{b} - \mathbf{proj}_W(\vec{b})\right\| \le \left\|\vec{b} - \vec{w}\right\| \quad for\ all\ \vec{w} \in W.$$

Example 6: From the previous example, the orthogonal projection of **b** on W = col(A) is

$$\mathbf{proj}_W(\vec{b}) = A\vec{x} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -5/33 \\ 20/11 \end{bmatrix} = \begin{bmatrix} -5/33 \\ -50/33 \\ 235/33 \end{bmatrix}.$$

Thus, we have obtained the best approximation to **b** in W.

What have we learned?

We have learned to :

1. Construct an orthonormal set of vectors from an orthogonal set of vectors.
2. Find the coordinate vector with respect to a given orthonormal basis.
3. Construct an orthogonal basis from a nonstandard basis in $\Re^n$ using the Gram-Schmidt process.
4. Find the least squares solution to a linear system A**x** = **b**.
5. Find the orthogonal projection on col(A).
6. Obtain the best approximation.

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

• Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition