MAC 2103

Module 2

Systems of Linear Equations and Matrices II

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Learning Objectives

Upon completing this module, you should be able to :

1. Find the inverse of a square matrix.
2. Determine whether a matrix is invertible.
3. Construct and identify elementary matrices; represent A and A^{-1} as a product of elementary matrices.
4. Solve systems of linear equations by using the inverse matrix.
5. Identify diagonal, triangular and symmetric matrices.

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Systems of Linear Equations and Matrices II

There are four major topics in this module:

Inverses
Elementary Matrices
Systems of Equations and Invertibility
Diagonal, Triangular, and Symmetric Matrices

Inverse of a Square Matrix

Let A represent a square matrix as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of matrix A can be obtained as follows:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

One important condition: $ad - bc \neq 0$

This will let us know whether the matrix is invertible or not.

What happens if *ad - bc = 0* ?

The matrix is not invertible; it has no inverse.

If matrix A has no inverse, then A is said to be singular.

Example: Finding an Inverse

Let A be a square matrix as follows:

$$A = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse of matrix A is:

$$A^{-1} = \frac{1}{(1)(6)-(-1)(-3)}\begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{6-3}\begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$$

Note:

The matrix is invertible because *ad - bc* produces a nonzero value.

How do we know the resulting matrix is the inverse of A?

Example: Finding an Inverse (Cont.)

How do we know the resulting matrix is the inverse of A?

$$A^{-1} = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}$$

Multiply the two matrices:

The product of a matrix and its inverse matrix is the identity matrix *I*. Notice that the inverse matrix of an inverse matrix is the original matrix.

$$AA^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix} \qquad A^{-1}A = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = (A^{-1})^{-1}A^{-1} \qquad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = A^{-1}(A^{-1})^{-1}$$

Can We Use the Gauss-Jordan Elimination Method to Find the Inverse?

The answer is YES.

This is actually a better method. However, the previous method is practical for a 2 x 2 matrix.

Let's use the Gauss-Jordan elimination method to find the inverse of matrix A; and identify how many elementary row operations that we need to produce the inverse matrix by converting *A* to *I*.

How?

1. Reduce A to the identity matrix *I* by elementary row operations.
2. Apply the same elementary row operations to the identity matrix *I* to produce the inverse.

Can We Use the Gauss-Jordan Elimination Method to Find the Inverse? (Cont.)

Step I: Adjoin the ***I*** to the right side of A.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -3 & 6 & 0 & 1 \end{array}\right] = [A \mid I]$$

Step II: Apply elementary row operations until the left side is reduced to the identity matrix *I* and the right side will be the inverse.

Label r1 (row 1) and r2 (row 2).

$$\begin{array}{c} r1 \\ r2 \end{array} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -3 & 6 & 0 & 1 \end{array}\right]$$

Step I: Adjoin the ***I*** to the right side of A as follows:

[*A* | *I*]

Step II: Apply elementary row operations until the left side is reduced to ***I*** and the right side will be the inverse.

[*I* | A^{-1}]

Can We Use the Gauss-Jordan Elimination Method to Find the Inverse? (Cont.)

We want a zero below the leading 1 at r1:

$$\begin{matrix} r1 \\ 3r1 + r2 \rightarrow r2 \end{matrix} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{array}\right]$$

We want a leading 1 at r2:

$$\begin{matrix} r1 \\ \frac{1}{3}r2 \rightarrow r2 \end{matrix} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{3} \end{array}\right]$$

We want a zero above the leading 1 at r2:

$$\begin{matrix} r2 + r1 \rightarrow r1 \\ r2 \end{matrix} \left[\begin{array}{cc|cc} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & 1 & \frac{1}{3} \end{array}\right]$$

We have just used the Gauss-Jordan elimination method to obtain the inverse; it takes three elementary row operations to convert *A* to *I*.

= [*I* | A^{-1}], $A^{-1} = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$

Elementary Row Operations and Elementary Matrices

By definition, an *n x n* matrix is called an elementary matrix if it can be obtained from the *n x n* identity matrix I_n by performing a single elementary row operation.

A *2 x 2* matrix is called an elementary matrix if it can be obtained from the *2 x 2* identity matrix I_2 by performing a single elementary row operation.

A *3 x 3* matrix is called an elementary matrix if it can be obtained from the *3 x 3* identity matrix I_3 by performing a single elementary row operation.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: $AI_n = A$ where *n* is the size in a *n x n* matrix.

If A is an *m x n* matrix, then we will have the following:

$$AI_n = A$$

$$I_m A = A$$

Elementary Row Operations and Elementary Matrices (Cont.)

If you remember from our previous problem, in the first elementary row operation, we add 3 times the first row to the second row.

$$\begin{array}{r} r1 \\ 3r1 + r2 \to r2 \end{array} \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

The corresponding first elementary matrix is obtained by adding 3 times the first row of I_2 to the second row. This is a row replacement for the second row.

$$\begin{array}{r} r1 \\ r2 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\begin{array}{r} r1 \\ 3r1 + r2 \to r2 \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = E_1$$

Elementary Row Operations and Elementary Matrices (Cont.)

In the second elementary row operation, we multiply the second row by 1/3.

$$\begin{array}{r} r1 \\ \frac{1}{3}r2 \to r2 \end{array} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

The corresponding second elementary matrix is obtained by multiplying the second row of I_2 by 1/3. This is a scaling of the second row.

$$\begin{array}{r} r1 \\ r2 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\begin{array}{r} r1 \\ \frac{1}{3}r2 \to r2 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = E_2$$

Theorem 1.5.1

If the elementary matrix *E* results from performing a certain row operation on I_m and if *A* is an $m \times n$ matrix, then *EA* is the matrix that results when this same row operation is performed on *A*.

Elementary Row Operations and Elementary Matrices (Cont.)

In the third elementary row operation, we add 1 times the second row to the first row.

$$\begin{matrix} r2+r1\to r1 \\ r2 \end{matrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The corresponding third elementary matrix is obtained by adding 1 times the second row of I_2 to the first row. This is a row replacement.

$$\begin{matrix} r1 \\ r2 \end{matrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I_2$$

$$\begin{matrix} r2+r1\to r1 \\ r2 \end{matrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}=E_3$$

So far, we have constructed three elementary matrices E_3, E_2 *and* E_1, which perform elementary row operations by multiplication on the left.

By multiplication of these elementary matrices, we obtain

$[E_3E_2E_1A \mid E_3E_2E_1I]$

$=[I \mid A^{-1}]$ from $[A \mid I]$.

Elementary Row Operations and Elementary Matrices (Cont.)

Based on the three elementary row operations performed in our previous problem, we can represent our Gauss-Jordan elimination as multiplications on the left by elementary matrices:

$$\begin{matrix} r1 \\ 3r1+r2\to r2 \end{matrix}\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{array}\right]=[E_1A \mid E_1I]$$

$$\begin{matrix} r1 \\ \frac{1}{3}r2\to r2 \end{matrix}\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{3} \end{array}\right]=[E_2E_1A \mid E_2E_1I]$$

$$\begin{matrix} r2+r1\to r1 \\ r2 \end{matrix}\left[\begin{array}{cc|cc} 1 & 0 & 2 & \frac{1}{3} \\ 0 & 1 & 1 & \frac{1}{3} \end{array}\right]=[E_3E_2E_1A \mid E_3E_2E_1I]=[I \mid A^{-1}]$$

How to Represent A^{-1} and A as a Product of Elementary Matrices?

Now, we can represent our inverse matrix as follows:

$A^{-1} = E_3E_2E_1I_2 = E_3E_2E_1$

Let's check it:

$$E_3E_2E_1 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix} = A^{-1}$$

How to Represent A^{-1} and A as a Product of Elementary Matrices? (Cont.)

Since $E_3E_2E_1A = I_2$ and

$A^{-1} = E_3E_2E_1I_2 = E_3E_2E_1$

we can obtain

$A = (A^{-1})^{-1} = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$

$$AA^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_3E_2E_1$$
$$= E_1^{-1}E_2^{-1}(I)E_2E_1 = E_1^{-1}E_2^{-1}E_2E_1$$
$$= E_1^{-1}(I)E_1 = E_1^{-1}E_1 = I$$

Thus, A can be represented as a product of elementary matrices since the inverse of a elementary matrix is an elementary matrix of the same type. There are three types, namely, scaling, rows interchange, and row replacement.

Theorem 1.5.3

If A is an $n \times n$ matrix, then the following statements are equivalent:

(a) A is invertible

(b) $A\vec{x} = \vec{0}$ has only the trivial solution

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

How to Represent A^{-1} and A as a Product of Elementary Matrices? (Cont.)

We know from our example that

$$E_3E_2E_1 = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = A^{-1}$$

Theorem 1.5.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Now, we see that $E_1^{-1}E_2^{-1}E_3^{-1} = \left(\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix} = A$$

where E_1^{-1}, E_2^{-1} and E_3^{-1} are all elementary matrices. It is an easy check to see that $E_j^{-1}E_j = I$ for j = 1, 2, and 3.

How to Solve a System of Linear Equations Using the Inverse Matrix?

Let's look at a system.

$$x_1 - x_2 = -1$$
$$-3x_1 + 6x_2 = 3$$

$$A = \begin{bmatrix} 1 & -1 \\ -3 & 6 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Theorem 1.6.2

If A is an invertible *n x n* matrix, then for each *n x 1* matrix $\vec{b}$, the system of equations $A\vec{x} = \vec{b}$ has exactly one solution, namely, $\vec{x} = A^{-1}\vec{b}$.

How to solve using the inverse matrix?

Step 1: Use the Gauss-Jordan elimination to solve for the inverse.

$$A^{-1} = \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$$

Remember: This was solved at the beginning.

How to Solve a System of Linear Equations Using the Inverse Matrix? (Cont.)

Step 2: Solve for $\vec{x}$ by using A^{-1} as in Theorem 1.6.2 as follows:

$$\vec{x} = A^{-1}\vec{b}$$

$$= \begin{bmatrix} 2 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(-1) + (\frac{1}{3})(3) \\ (1)(-1) + (\frac{1}{3})(3) \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus, $x_1 = -1$ and $x_2 = 0$

How to Identify a Diagonal Matrix?

Diagonal Matrix: A square matrix in which all the entries off the main diagonal are zero.

Examples:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -7 \end{bmatrix} \quad \begin{bmatrix} 10 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note:

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero.

Can you identify which one of the examples is noninvertible?

How to Identify a Triangular Matrix?

Upper Triangular Matrix: A square matrix in which all the entries below the main diagonal are zero.

Lower Triangular Matrix: A square matrix in which all the entries above the main diagonal are zero.

Triangular Matrix: A square matrix that is either an upper triangular matrix or a lower triangular matrix.

Note:

A triangular matrix is invertible if and only if all of its diagonal entries are nonzero - just like the diagonal matrix.

Examples:

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Upper Triangular Matrix   Lower Triangular Matrix

Recall: Transpose of a Matrix

Transpose of a matrix: If *A* is any *m x n* matrix, then the transpose, denoted by A^T, is defined to be the *n x m* matrix that results from interchanging the rows and columns of A.

Example:

$$A = \left[a_{ij}\right] = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 5 \\ 8 & 9 \end{bmatrix} \qquad A^T = \left[a_{ji}\right] = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 9 \end{bmatrix}$$

4 x 2   2 x 4

How to Identify a Symmetric Matrix?

Symmetric Matrix: A square matrix A is called symmetric if $A = A^T$.

Examples:

$$\begin{bmatrix} 1 & -2 & 5 \\ -2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Theorem 1.7.2

If A and B are symmetric matrices with the same size, and if s is a scalar, then:

1. A^T is symmetric.

2. $A + B$ and $A - B$ are symmetric.

3. sA is symmetric.

Theorem 1.7.3

If A is an invertible symmetric matrix, than A^{-1} is symmetric.

What Have We Learned?

We have learned to:

1. Find the inverse of a square matrix.
2. Determine whether a matrix is invertible.
3. Construct and identify elementary matrices; represent A and A^{-1} as a product of elementary matrices.
4. Solve systems of linear equations by using the inverse matrix.
5. Identify diagonal, triangular and symmetric matrices.

Credit

Some definitions and theorems have been adapted/modified in part/whole from the following textbook:

•Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition