MAC 2103

Module 4

Vectors in 2-Space and 3-Space I

Learning Objectives

In this module, we apply our earlier ideas specifically to vectors in 2-space, $\Re^2$, (in the xy-plane) in two dimensions and to vectors in 3-space, $\Re^3$,(in the xyz-space) in three dimensions.

Learning Objectives (Cont.)

Upon completing this module, you should be able to:

1. Determine the components of a vector in $\Re^2$ and $\Re^3$.
2. Perform vector addition, subtraction, and scalar multiplication in $\Re^2$ and $\Re^3$.
3. Find the norm of a vector and the distance between points in $\Re^2$ and $\Re^3$.
4. Find the dot product of two vectors in $\Re^2$ and $\Re^3$.
5. Use the dot product to find the angle between two vectors in $\Re^2$ and $\Re^3$.
6. Find the projection of a vector onto another vector in $\Re^2$ and $\Re^3$, and express the original vector as a sum of two orthogonal vectors.
7. Find the distance between a point and a line in $\Re^2$ and $\Re^3$.

Vectors in $\Re^2$ and $\Re^3$

There are three major topics in this module:

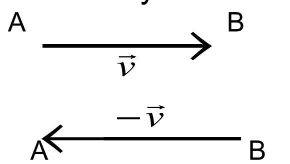
Introduction to Vectors (Geometric)

Norm of a Vector; Vector Operations

Dot Product; Projections

What are Vectors in $\Re^2$ and $\Re^3$?

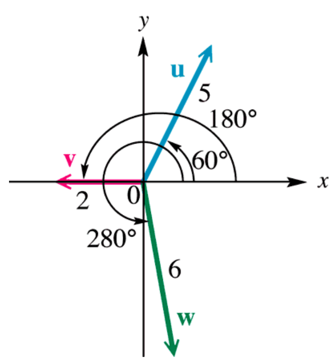
- Vectors can be represented as directed line segments or arrows in $\Re^2$ and $\Re^3$.
- The direction of the arrow specifies the direction of the vector.
- A vector that starts from an initial point A and terminates at a point B can be represented as $\overrightarrow{AB}$.
- A vector is usually denoted in lowercase boldface type (like **v**) in the textbook or with an arrow above it when we write it by hand. For example: $\mathbf{v} = \vec{v} = \overrightarrow{AB}$.



What are Vectors in $\Re^2$ and $\Re^3$? (Cont.)

- The magnitude of the vector is the length of the vector.
- The vector of length zero is called the zero vector.
- Vectors with the same magnitude and same direction are equal to each other.
- A vector **v** in standard position has its starting point at the origin. The coordinates (v_1, v_2) of the terminal point of **v** are called the components of **v**.

$$\mathbf{v} = \vec{v} = (v_1, v_2)$$



Note: The negative of vector **v** is defined to be the vector that has the same magnitude as **v** but is oppositely directed.

What are Vectors in $\Re^2$ and $\Re^3$? (Cont.)

If *s* is any scalar, then a vector of the form *s***v** is called a scalar multiple of **v**.

$$s\mathbf{v} = s\vec{v} = s(v_1, v_2) = (sv_1, sv_2)$$

For example, if **v** = (2,-7) and *s* =- 5, then

$$-5\vec{v} = -5(v_1, v_2) = (-5v_1, -5v_2) = (-10, 35)$$

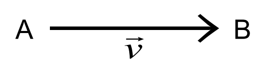
What are Vectors in $\Re^2$ and $\Re^3$? (Cont.)

- If **v** and **u** are any two vectors in standard position, then the sum and difference of the two vectors is also a vector. It's also a vector in standard position.

$$\mathbf{v} + \mathbf{u} = \vec{v} + \vec{u} = (v_1, v_2) + (u_1, u_2) = (v_1 + u_1, v_2 + u_2)$$

$$\mathbf{v} - \mathbf{u} = \vec{v} - \vec{u} = (v_1, v_2) - (u_1, u_2) = (v_1 - u_1, v_2 - u_2)$$

What are the Components of a Vector in $\Re^3$?



If the initial point of $\overrightarrow{AB}$ is A(x_1,y_1,z_1) and the terminal point of $\overrightarrow{AB}$ is B(x_2,y_2,z_2) in $\Re^3$, then the components of $\overrightarrow{AB}$ can be obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point.

$$\mathbf{v} = \vec{v} = \overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Example: Suppose the initial point of $\overrightarrow{AB}$ is A(1,-2,5) and terminal point is B(-1,4,9), then the components of the vector $\mathbf{v} = \vec{v} = \overrightarrow{AB} = (-2,6,4)$. We see that the vector $\overrightarrow{AB}$ is equal to the vector **v** in standard position.

Example

Suppose $\vec{u} = (-5,1,6)$

$\vec{v} = (1,0,-8)$

Find the components of $7\vec{u} - 2\vec{v}$.

$$7\vec{u} - 2\vec{v} = 7\vec{u} + (-2)\vec{v} = 7(-5,1,6) + (-2)(1,0,-8)$$
$$= ((7)(-5) + (-2)(1), (7)(1) + (-2)(0), (7)(6) + (-2)(-8))$$
$$= (-37,7,58)$$

Note: In chapter 1, we would represent these vectors as column matrices:

$$\vec{u} = \begin{bmatrix} -5 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 6 \end{bmatrix}^T \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \end{bmatrix}^T$$

Some Important Properties of a Vector Space

If **u**, **v**, and **w** are vectors in $\Re^2$, $\Re^3$, or any vector space and k and s are scalars, then the following hold:

a) **u** + **v** = **v** + **u**

b) (**u** + **v**) + **w** = **u** + (**v** + **w**)

c) **u** + **0** = **0** + **u** = **u**

d) **u** + **(-u)** = **0**

e) k(s**u**)= ks(**u**)

f) k(**u + v**)= k**u** + k**v**

g) (k + s) **u** = k**u** + s**v**

h) *1***u** = **u**

What is the Norm of a Vector in $\Re^3$?

- The norm of a vector **u,** $\|\vec{u}\| = \|\mathbf{u}\|$**,** is the length or the magnitude of the vector **u**.
- If **u** = (u_1, u_2, u_3) = (-1, 4, -8), then the norm of the vector **u** is

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{(-1)^2 + 4^2 + (-8)^2} = 9$$

- This is just the distance of the terminal point to the origin for **u** in standard position.

Note: If **u** is any nonzero vector, then $\dfrac{\mathbf{u}}{\|\vec{u}\|}$ is a unit vector. A unit vector is a vector of norm 1.

How to Find the Distance Between Two Points?

- If A(x_1,y_1,z_1) and B(x_2,y_2,z_2) are two points in $\Re^3$, then the distance between the two points is the length, the magnitude, and the norm of the vector $\overrightarrow{AB}$.

$$d = \left\|\overline{AB}\right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

How to Find Dot Product of Two Vectors in Terms of the Components of the Vectors?

If **u** = (u_1, u_2, u_3) and **v** = (v_1, v_2, v_3) , then the dot product of the two vectors in terms of the components of the vectors is:

$$\mathbf{u} \cdot \mathbf{v} = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \vec{u}\vec{v}^T$$

Example: If **u** = (*3, 0, -1*) and **v** = (*2, 9, -2*) , then the dot product of the two vectors is:

$$\mathbf{u} \cdot \mathbf{v} = \vec{u} \cdot \vec{v} = (3)(2) + (0)(9) + (-1)(-2) = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 9 & -2 \end{bmatrix}$$

$$= \vec{u}\vec{v}^T = 6 + 0 + 2 = 8$$

How to Find the Angle Between Vectors?

By definition, if **u** and **v** are nonzero vectors in $\Re^2$ and $\Re^3$ and θ is the angle between **u** and **v**, then the dot product of the two vectors is:

$$\mathbf{u} \cdot \mathbf{v} = \vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

Thus, if **u** and **v** are nonzero vectors, the angle can be obtained by:

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Note: From the previous slide,

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{8}{\sqrt{10}\sqrt{89}} \quad .$$

Some Important Properties of the Dot Product

If **u**, **v**, and **w** are vectors in $\Re^2$ and $\Re^3$ and *s* is a scalar, then the following relationships hold:

a) **u** · **v** = **v** · **u**

b) **u** · (**v** + **w**) = **u** · **v** + **u** · **w**

c) *s* (**u** · **v**) = (*s* **u**) · **v** = **u** · (s **v**)

d) $\mathbf{v} \cdot \mathbf{v} = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ and $\|\vec{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

e) $\mathbf{v} \cdot \mathbf{v} = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2 > 0$ if and only if **v** ≠ **0**, and **v** · **v** = 0 iff **v** = **0**

If the vectors **u** and **v** are nonzero and θ is the angle between them, then θ = π/2 if and only if **u**·**v** = 0. Then,

u and **v** are perpendicular or orthogonal.

How to Find the Projection of a Vector onto Another Vector?

If **u** and **v** are vectors in in $\Re^2$ and $\Re^3$ and if **a** ≠ **0**, then

$$proj_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \mathbf{u}\cdot\frac{\mathbf{a}}{\|\mathbf{a}\|}\frac{\mathbf{a}}{\|\mathbf{a}\|}$$ (vector component of **u** along **a**)

$$\mathbf{u} - proj_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$ (vector component of **u** orthogonal or perpendicular to **a**)

Thus, the $proj_{\mathbf{a}}\mathbf{u}$ and **u** - $proj_{\mathbf{a}}\mathbf{u}$ are orthogonal vectors whose sum is **u**. The dot product of $proj_{\mathbf{a}}\mathbf{u}$ and **u** - $proj_{\mathbf{a}}\mathbf{u}$ is zero**.**

$$(proj_{\mathbf{a}}\mathbf{u})\cdot(\mathbf{u} - proj_{\mathbf{a}}\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}\right)\cdot\left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}\right) =$$

$$\frac{(\mathbf{u}\cdot\mathbf{a})^2}{\|\mathbf{a}\|^2} - \frac{(\mathbf{u}\cdot\mathbf{a})^2(\mathbf{a}\cdot\mathbf{a})}{\|\mathbf{a}\|^2\|\mathbf{a}\|^2} = \frac{(\mathbf{u}\cdot\mathbf{a})^2}{\|\mathbf{a}\|^2} - \frac{(\mathbf{u}\cdot\mathbf{a})^2}{\|\mathbf{a}\|^2} = 0$$

How to Find the Projection of a Vector onto Another Vector and Express the Original Vector as the Sum of Two Orthogonal Vectors?

Example

Let **u** = (3,1,-7) and **a** = (1,0,5). Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a**.

Solution:

Step 1: Find the dot product of the two vectors.

$$\mathbf{u}\cdot\mathbf{a} = \vec{u}\cdot\vec{a} = (3)(1)+(1)(0)+(-7)(5) = 3+0+(-35) = -32$$

Step 2: Find the norm of **a**.

$$\|\mathbf{a}\| = \|\vec{a}\| = \sqrt{(1)^2+(0)^2+(5)^2} = \sqrt{1+0+25} = \sqrt{26}$$

How to Find the Projection of a Vector onto Another Vector? (Cont.)

Step 3: Solve for the vector component of **u** along **a**.

$$proj_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{-32}{26}\mathbf{a} = \frac{-16}{13}(1,0,5) = (\frac{-16}{13},0,\frac{-80}{13})$$

Step 4: Solve for the vector component of **u** orthogonal to **a**.

$$\mathbf{u} - proj_{\mathbf{a}}\mathbf{u} = (3,1,-7) - (\frac{-16}{13},0,\frac{-80}{13}) = (\frac{55}{13},1,\frac{-11}{13})$$

Note: $(\mathbf{u} - proj_{\mathbf{a}}\mathbf{u}) + proj_{\mathbf{a}}\mathbf{u} = (\frac{55}{13},1,\frac{-11}{13}) + (\frac{-16}{13},0,\frac{-80}{13})$

$$= (\frac{39}{13},1,\frac{-91}{13}) = (3,1,-7) = \mathbf{u}$$

How to Find the Projection of a Vector onto Another Vector? (Cont.)

Step 5: Check to see if the two component vectors are orthogonal.

$$proj_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{-32}{26}\mathbf{a} = \frac{-16}{13}(1,0,5) = (\frac{-16}{13},0,\frac{-80}{13})$$

$$\mathbf{u} - proj_{\mathbf{a}}\mathbf{u} = (3,1,-7) - (\frac{-16}{13},0,\frac{-80}{13}) = (\frac{55}{13},1,\frac{-11}{13})$$

$$(proj_{\mathbf{a}}\mathbf{u})\cdot(\mathbf{u} - proj_{\mathbf{a}}\mathbf{u}) = (\frac{-16}{13},0,\frac{-80}{13})\cdot(\frac{55}{13},1,\frac{-11}{13})$$

$$= \frac{-880}{169} + 0 + \frac{880}{169} = 0$$

How to Find the Distance Between a Point and a Line?

Example

Find the distance *D* from the point (-3,1) to the line 4x+3y+4=0.

Solution:

We can use the distance formula in Equation (13)

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

to find the distance *D*. In our problem, $x_0=-3$, $y_0=1$, a=4, b=3, and c=4.

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} = \frac{|(4)(-3) + (3)(1) + 4|}{\sqrt{4^2 + 3^2}} = \frac{|-5|}{5} = \frac{5}{5} = 1$$

What have we learned?

We have learned to:

1. Determine the components of a vector in $\Re^2$ and $\Re^3$.
2. Perform vector addition, subtraction, and scalar multiplication in $\Re^2$ and $\Re^3$.
3. Find the norm of a vector and the distance between points in $\Re^2$ and $\Re^3$.
4. Find the dot product of two vectors in $\Re^2$ and $\Re^3$.
5. Use the dot product to find the angle between two vectors in $\Re^2$ and $\Re^3$.
6. Find the projection of a vector onto another vector in $\Re^2$ and $\Re^3$, and express the original vector as a sum of two orthogonal vectors.
7. Find the distance between a point and a line in $\Re^2$ and $\Re^3$.

Credit

Some of these slides have been adapted/modified in part/whole from the text or slides of the following textbooks:

- Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition
- Margaret L. Lial, John Hornsby, David I. Schneider, Trigonometry, 8th Edition