MAC 2103

Module 5

Vectors in 2-Space and 3-Space II

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Learning Objectives

Upon completing this module, you should be able to:

1. Determine the cross product of a vector in $\Re^3$.
2. Determine a scalar triple product of three vectors in $\Re^3$.
3. Find the area of a parallelogram and the volume of a parallelepiped in $\Re^3$.
4. Find the sine of the angle between two vectors in $\Re^3$.
5. Find the equation of a plane in $\Re^3$.
6. Find the parametric equations of a line in $\Re^3$.
7. Find the distance between a point and a plane in $\Re^3$.

Rev.F09

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Vectors in 2-Space and 3-Space II

There are two major topics in this module:

Cross Products
Lines and Planes in 3-Space

Quick Review: The Norm of a Vector in $\Re^3$

- The norm of a vector **u**, $\|\vec{u}\| = \|\mathbf{u}\|$, is the length or the magnitude of the vector **u**.
- If **u** = (u_1, u_2, u_3) = (-1, 4, -8), then the norm of the vector **u** is

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{(-1)^2 + 4^2 + (-8)^2} = 9$$

- This is just the distance of the terminal point to the origin for **u** in standard position.

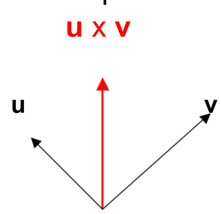
Note: If **u** is any nonzero vector, then $\dfrac{\mathbf{u}}{\|\vec{u}\|}$ is a unit vector. A unit vector is a vector of norm 1.

The Cross Product of Two Vectors in $\Re^3$

- The cross product of two vectors **u** = (u_1,u_2,u_3) and **v** = (v_1,v_2,v_3), **u** x **v**, in $\Re^3$ is a vector in $\Re^3$.

- The direction of the cross product, **u** x **v**, is always perpendicular to the two vectors **u** and **v** and the plane determined by **u** and **v** that is parallel to both **u** and **v**.

- The norm of the cross product is $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin(\theta).$



The Cross Product of Two Vectors in $\Re^3$ (Cont.)

The cross product can be represented symbolically in the form of a 3 x 3 determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

where **i** = (1,0,0), **j** = (0,1,0), **k** = (0,0,1) are standard unit vectors.

Note: Every vector in $\Re^3$ is expressible in terms of the standard unit vectors.

v = $(v_1,v_2,v_3) = v_1(1,0,0) + v_2\,(0,1,0) + v_3\,(0,0,1) = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$

The Cross Product of Two Vectors in $\Re^3$ (Cont.)

Example: Find the cross product of **u** = (0,2,-3) and **v** = (2,6,7).

Solution:

The cross product can be obtained as follows:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix}\mathbf{k}$$

$$= 32\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$$

$$= (32, -6, -4)$$

To check the orthogonality:

u · (**u** x **v**) = (0,2,-3) · (32, -6,-4)=(0)(32)+(2)(-6)+(-3)(-4)=0

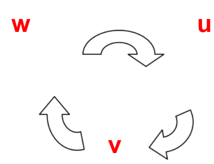
v · (**u** x **v**) = (2,6,7) · (32, -6,-4)=(2)(32)+(6)(-6)+(7)(-4)=0

How to Find a Scalar Triple Product?

First of all, what is a scalar triple product?

A scalar triple product of three vectors is a combination of a dot product with a cross product as follows:

u · (**v** x **w**) = **v** · (**w** x **u**) = **w** · (**u** x **v**)



How to Find a Scalar Triple Product? (Cont.)

Example: Calculate a scalar triple product of **u** = (0,2,-3), **v** = (2,6,7), and **w** = (-1,0,3).

Solution:

u · (**v** x **w**) = **v** · (**w** x **u**) = **w** · (**u** x **v**)

From the previous slide, **u** x **v** = (32,-6,-4). Similarly, we can obtain **u** x **w** = (6,3,2) and **v** x **w** = (18,-13,6). Try it now. All three should give the same outcome or result.

w · (**u** x **v**) = (-1,0,3) · (32, -6,-4)=(-1)(32)+(0)(-6)+(3)(-4)= -44
v · (**w** x **u**) = (2,6,7) · (-6,-3,-2)=(2)(6)+(6)(3)+(7)(2)=-44
u · (**v** x **w**) = (0,2,-3) · (18, -13,6)=(0)(18)+(2)(-13)+(-3)(6)=-44

How to Find the Area of a Parallelogram?

The area of the parallelogram, *A*, determined by **u** = (u_1,u_2,u_3) and **v** = (v_1,v_2,v_3) in $\Re^3$ is as follows:

$$A = \|\vec{u} \times \vec{v}\|$$

How to Find the Area of a Parallelogram? (Cont.)

Example: Find the area of the parallelogram of **u** = (1,-1,2) and **v** = (0,3,1).

Solution:

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}\vec{k}$$

$$= -7\vec{i} - \vec{j} + 3\vec{k} = (-7,-1,3)$$

Thus, the area of the parallelogram is

$$A = \|\vec{u}\times\vec{v}\| = \sqrt{(-7)^2 + (-1)^2 + (3)^2} = \sqrt{59}$$

How to Find the Volume of a Parallelepiped?

The volume of a parallelepiped, V, determined by **u** = (u_1,u_2,u_3), **v** = (v_1,v_2,v_3), and **w** = (w_1,w_2,w_3) in $\Re^3$ is as follows:

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

How to Find the Volume of a Parallelepiped? (Cont.)

Example: Find the volume of the parallelepiped of **u** = (-1,-2,4), **v** = (4,5,1), and **w** = (1,2,4).

Solution:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 4 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= -1\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + 4\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}$$

$$= -1(18) - 2(15) + 4(3) = -36$$

Thus, the volume of the parallelepiped is

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = |-36| = 36$$

Note: If **u**, **v**, and **w** have the same initial point, then they lie in the same plane and the volume of the parallelepiped is zero.

How to Find the Sine of the Angle Between Two Vectors?

Example: Use the cross product to find the sine of the angle between the vectors **u** = (2,3,-6) and **v** = (2,3,6).

Solution:

We can use equation (6) derived from the Lagrange's identity.

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin(\theta)$$

$$\sin(\theta) = \frac{\|\vec{u}\times\vec{v}\|}{\|\vec{u}\|\|\vec{v}\|}$$

Note 1: Lagrange's identity: ||**u** x **v**||^2 = ||**u**||^2 ||**v**||^2 - (**u**·**v**)^2.

Note 2: If θ denotes the angle between two nonzero vectors **u** and **v**, then **u**·**v** = ||**u**|| ||**v**|| cos(θ). (From previous module.)

How to Find the Sine of the Angle Between Two Vectors? (Cont.)

Step 1: Find the cross product of **u** and **v**.

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & -6\\ 2 & 3 & 6\end{vmatrix}=\begin{vmatrix}3 & -6\\ 3 & 6\end{vmatrix}\vec{i}-\begin{vmatrix}2 & -6\\ 2 & 6\end{vmatrix}\vec{j}+\begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix}\vec{k}$$

$$=36\vec{i}-24\vec{j}+0\vec{k}=(36,-24,0)$$

Step 2: Find the norm of the cross product of **u** and **v**.

$$\|\vec{u}\times\vec{v}\|=\sqrt{(36)^2+(-24)^2+(0)^2}=\sqrt{1872}$$

How to Find the Sine of the Angle Between Two Vectors? (Cont.)

Step 3: Find sin(θ).

$$\sin(\theta)=\frac{\|\vec{u}\times\vec{v}\|}{\|\vec{u}\|\|\vec{v}\|}$$

$$=\frac{\sqrt{1872}}{\sqrt{(2)^2+(3)^2+(-6)^2}\sqrt{(2)^2+(3)^2+(6)^2}}$$

$$=\frac{\sqrt{1872}}{\sqrt{49}\sqrt{49}}=\frac{\sqrt{1872}}{49}$$

Some Important Relationships Involving Cross Products and Dot Products

If **u**, **v**, and **w** are vectors in $\Re^3$, then the following hold:

a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to **u**)

b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to **v**)

c) $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2\, ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (Lagrange's Identity)

Some Important Properties of Cross Products

If **u**, **v**, and **w** are vectors in $\Re^3$ and *s* is a scalar, then the following hold: (See Theorem 3.4.2)

a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

d) $s(\mathbf{u} \times \mathbf{v}) = (s\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (s\mathbf{v})$

e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Quick Review: Finding the Distance Between Two Points in $\Re^3$?

If A(x_1,y_1,z_1) and B(x_2,y_2,z_2) are two points in $\Re^3$, then the distance between the two points is the length, the magnitude, and the norm of the vector $\overrightarrow{AB}$.

$$d = \left\|\overrightarrow{AB}\right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The Point-Normal Form and the Standard Form of the Equation of a Plane

To find the equation of the plane passing through the point $P_0(x_0,y_0,z_0)$ and having a nonzero vector **n** = (a, b, c) that is perpendicular (normal) to the plane, we proceed as follows:

a. Let $P(x,y,z)$ be any point in the plane but not equal to P_0.

b. Then the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is in the plane and parallel to the plane and orthogonal to **n**.

The Point-Normal Form and the Standard Form of the Equation of a Plane (Cont.)

Thus, $\vec{n}\cdot\overrightarrow{P_0P} = a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$ax+by+cz = ax_0+by_0+cz_0 = d$$

$$ax+by+cz=d$$

This is the standard linear equation of a plane in $\Re^3$.

Note if c = 0, we have the standard linear equation of a line in $\Re^2$.

The top equation is the so called point-normal form of the equation of a plane.

The Point-Normal Form and the Standard Form of the Equation of a Plane (Cont.)

Example: Find the equation of the plane passing through the point $P_0(-5,3,-2)$ and having a normal vector **n** = (-7, 2, 3).

Solution:

$$\vec{n}\cdot\overrightarrow{P_0P} = 0$$

$$-7(x-(-5))+2(y-3)+3(z-(-2))=0$$

$$-7(x+5)+2(y-3)+3(z+2)=0$$

$$-7x+2y+3z=35$$

We obtain the point-normal form of the equation of the plane and the standard form of the equation of the plane. Is P_0 in the plane? Does P_0 satisfy the two equations?

How to Find an Equation of a Plane Passing Through the Given Points in $\Re^3$?

P(-4,-1,-1), Q(-2,0,1) and R(-1,-2,-3) are in $\Re^3$, find an equation of the plane passing through these three points.

Solution:

Step 1: Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$$\overrightarrow{PQ} = (2,1,2) \qquad \overrightarrow{PR} = (3,-1,-2)$$

Step 2: Find the cross product

$$\overrightarrow{PQ} \times \overrightarrow{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \vec{k}$$

$$= 0\vec{i} + 10\vec{j} - 5\vec{k} = (0,10,-5) = \vec{n}$$

Recall: The direction of the cross product, **u** x **v**, is always perpendicular to the two vectors **u** and **v**.

How to Find an Equation of a Plane Passing Through the Given Points in $\Re^3$? (Cont.)

Step 3: Find the equation of the plane passing through the point *P*(*-4,-1,-1*) with the normal vector **n** = (0, 10, -5).

$$0(x+4)+10(y+1)+(-5)(z+1)=0$$
$$10y+10-5z-5=0$$
$$10y-5z=-5$$
$$2y-z=-1$$

Try this with point *Q* or point *R*.

How to Find the Parametric Equations of a Line Passing Through the Given Points?

Example: Find the parametric equations for the line through: $P_0(5,-2,4)$ and $P_1(7,2,-4)$.

Step 1: Find the vector $\vec{v} = (a,b,c) = \overrightarrow{P_0P_1}$

$$\overrightarrow{P_0P_1} = (2,4,-8)$$

Step 2: By using this vector and the point $P_0(5,-2,4)$, we can get the parametric equations as follows:

$$(x-5, y+2, z-4) = t(2,4,-8), t \in (-\infty,\infty)$$

or $x = 5+2t, y = -2+4t, z = 4-8t, t \in (-\infty,\infty)$

which are the component equations for $\vec{r} = \vec{r}_0 + t\vec{v}$ the vector equation form.

Note: The parametric equations for the line passing through the point $P(x_0,y_0,z_0)$ and parallel to a nonzero vector **v**=(a,b,c) are given by $(x-x_0,y-y_0,z-z_0) = t\mathbf{v}$, or in vector form: $\vec{r} - \vec{r}_0 = t\vec{v}$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

How to Find the Distance Between a Point and a Plane?

Example

Find the distance *D* from the point (-3,1,2) to the plane 2x+3y-6z+4=0.

Solution:

We can use the distance formula in Equation (9)

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

to find the distance *D*. In our problem, x_0=-3, y_0=1, z_0=2, a=2, b=3, c=-6, and d=4.

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|(2)(-3)+(3)(1)+(-6)(2)+4|}{\sqrt{2^2+3^2+(-6)^2}} = \frac{|-11|}{\sqrt{49}} = \frac{11}{7}$$

What have we learned?

We have learned to:

1. Determine the cross product of a vector in $\Re^3$.
2. Determine a scalar triple product of three vectors in $\Re^3$.
3. Find the area of a parallelogram and the volume of a parallelepiped in $\Re^3$.
4. Find the sine of the angle between two vectors in $\Re^3$.
5. Find the equation of a plane in $\Re^3$.
6. Find the parametric equations of a line in $\Re^3$.
7. Find the distance between a point and a plane in $\Re^3$.

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

• Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition