MAC 2103

Module 6

Euclidean Vector Spaces I

1

Learning Objectives

Upon completing this module, you should be able to:

1. Use vector notation in $\Re^n$.
2. Find the inner product of two vectors in $\Re^n$.
3. Find the norm of a vector and the distance between two vectors in $\Re^n$.
4. Express a linear system in $\Re^n$ in dot product form.
5. Find the standard matrix of a linear transformation from $\Re^n$ to $\Re^m$.
6. Use linear transformations such as reflections, projections, and rotations.
7. Use the composition of two or more linear transformations .

Rev.F09

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Euclidean Vector Spaces I

There are two major topics in this module:

Euclidean *n*-Space, $\Re^n$

Linear Transformations from $\Re^n$ to $\Re^m$

Some Important Properties of Vector Operations in $\Re^n$

If **u**, **v**, and **w** are vectors in $\Re^n$ and *k* and *s* are scalars, then the following hold: (See Theorem 4.1.1)

a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

d) $\mathbf{u} + (\mathbf{-u}) = \mathbf{0}$

e) $k(s\mathbf{u}) = (ks)\mathbf{u}$

f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

g) $(k + s)\mathbf{u} = k\mathbf{u} + s\mathbf{u}$

h) $1\mathbf{u} = \mathbf{u}$

Basic Vector Operations in $\Re^n$

Two vectors **u** = (u_1, u_2, ... , u_n) and **v** = (v_1, v_2, ... , v_n) are equal if and only if

$$u_1 = v_1 ,\ u_2 = v_2 ,\ \ldots ,\ u_n = v_n .$$

Thus,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 ,\ u_2 + v_2, \ldots,\ u_n + v_n)$$

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1 ,\ u_2 - v_2, \ldots,\ u_n - v_n)$$

and

$$5\mathbf{v} - 2\mathbf{u} = (5u_1 - 2v_1,\ 5u_2 - 2v_2, \ldots,\ 5u_n - 2v_n)$$

How to Find the Inner Product of Two Vectors in $\Re^n$?

- The inner product of two vectors **u** = (u_1,u_2,…,u_n) and **v** = (v_1,v_2,…,v_n), **u · v,** in $\Re^n$ is also known as the Euclidean inner product or dot product.
- The inner product, **u · v**, can be computed as follows:

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots + u_n \cdot v_n = \vec{u}^T \vec{v}$$

Example: Find the Euclidean inner product of **u** and **v** in $\Re^4$, if **u** = (2, -3, 6, 1) and **v** = (1, 9, -2, 4).

Solution:

$$\vec{u} \cdot \vec{v} = (2)(1) + (-3)(9) + (6)(-2) + (1)(4) = -33$$

$$= \begin{bmatrix} 2 & -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ -2 \\ 4 \end{bmatrix}$$

How to Find the Norm of a Vector in $\Re^n$?

As we have learned in a previous module, the norm of a vector in $\Re^2$ and $\Re^3$ can be obtained by taking the square root of the sum of square of the components as follows:

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}, \vec{u} = (u_1, u_2) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}, \vec{u} = (u_1, u_2, u_3) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

How to Find the Norm of a Vector in $\Re^n$? (Cont.)

Similarly, the Euclidean norm of **u** = (u_1,u_2,…,u_n), ||**u**||, in $\Re^n$ can be computed as follows:

Example: Find the Euclidean norm of **u** = (2, -3, 6, 1) in $\Re^4$.
Solution:

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}, \vec{u} = (u_1, u_2, \ldots, u_n)$$

$$\|\vec{u}\| = \sqrt{2^2 + (-3)^2 + 6^2 + 1^2} = \sqrt{4 + 9 + 36 + 1} = \sqrt{50} = 5\sqrt{2}$$

How to Find the Distance Between Two Vectors in $\Re^n$?

• The distance between **u** = $(u_1,u_2,\ldots,u_n)$ and **v** = $(v_1,v_2,\ldots,v_n)$ in $\Re^n$, d(**u**,**v**)**,** is also known as the Euclidean distance.

• The Euclidean distance, d(**u**,**v**)**,** can be computed as follows:

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\ldots+(u_n-v_n)^2}$$

Example: Suppose **u** = (2, -3, 6, 1) and **v** = (1, 9, -2, 4). Find the Euclidean distance between **u** and **v** in $\Re^4$,

Solution:

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{(2-1)^2+((-3)-9)^2+(6-(-2))^2+(1-4)^2}$$

$$= \sqrt{1+144+64+9} = \sqrt{218}$$

How to Express a Linear System in $\Re^n$ in Dot Product Form?

Example: Express the following linear system in dot product form.

$$x_1 - 5x_2 + 2x_3 + 9x_4 = 2$$
$$3x_1 + 2x_2 + 6x_3 - 2x_4 = 5$$
$$-4x_1 + x_2 + 2x_4 = -1$$
$$8x_1 - 1x_2 + 3x_3 - 7x_4 = 0$$

Solution:

$$\begin{bmatrix} (1,-5,2,9)\cdot(x_1,x_2,x_3,x_4) \\ (3,2,6,-2)\cdot(x_1,x_2,x_3,x_4) \\ (-4,1,0,2)\cdot(x_1,x_2,x_3,x_4) \\ (8,-1,3,-7)\cdot(x_1,x_2,x_3,x_4) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \\ 0 \end{bmatrix} \Leftrightarrow [\vec{a}_i \cdot \vec{x}] = \vec{b}, i = 1,2,3,4$$

How to Express a Linear Transformation from $\Re^3$ to $\Re^4$ in Matrix Form?

The linear transformation T: $\Re^3 \rightarrow \Re^4$ defined by the equations

$$w_1 = 2x_1 - x_2 + x_3$$
$$w_2 = x_1 + 8x_2 - 3x_3$$
$$w_3 = -x_1 + 2x_2 - 2x_3$$
$$w_4 = 6x_1 - x_2 + 2x_3$$

can be expressed in matrix form as follows:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 8 & -3 \\ -1 & 2 & -2 \\ 6 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Leftrightarrow \vec{w} = A\vec{x}, w_i = \vec{a}_i^T \vec{x} = \vec{a}_i \cdot \vec{x}, i = 1,2,3,4$$

What is the Standard Matrix for a Linear Transformation?

Based on our example in previous slide, the standard matrix can be found from the linear transformation T: $\Re^3 \rightarrow \Re^4$ expressed in matrix form.

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 8 & -3 \\ -1 & 2 & -2 \\ 6 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Leftrightarrow \vec{w} = A\vec{x}$$

The standard matrix for T is:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 8 & -3 \\ -1 & 2 & -2 \\ 6 & -1 & 2 \end{bmatrix}$$

Example and Notations

Example: Find the standard matrix for the linear transformation T defined by the formula as follows:

$$T(x_1, x_2) = (3x_1 - 2x_2, 7x_1 + x_2) = (w_1, w_2)$$

Solution: In this case, the linear operator T assigns a unique point (w_1, w_2) in $\Re^2$ to each point (x_1, x_2) in $\Re^2$ according to the rule

$$(w_1, w_2) = (3x_1 - 2x_2, 7x_1 + x_2),$$

or as a linear system, it is as follows:

$$w_1 = 3x_1 - 2x_2$$
$$w_2 = 7x_1 + x_2$$

Note:

A linear transformation

T: $\Re^n \rightarrow \Re^m$

is also known as a linear operator.

Example and Notations (Cont.)

A linear system can be expressed in matrix form.

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \vec{w} = A\vec{x} = T_A(\vec{x}) = T(\vec{x}) = [T]\vec{x}$$

In this case, the standard matrix for T is

$$[T] = [T_A] = A = \begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix}$$

In general, the linear transformation is represented by T: $\Re^n \rightarrow \Re^m$ or T_A: $\Re^n \rightarrow \Re^m$; the matrix $A = [a_{ij}]$ is called the standard matrix for the linear transformation, and T is called multiplication by A.

Zero Transformation and Identity Operator

If *0* is the m x n zero matrix, then for every vector **x** in $\Re^n$, we will have the zero transformation from $\Re^n$ to $\Re^m$, T_0: $\Re^n \to \Re^m$, where T_0 is called multiplication by *0*.

$$T_0(\vec{x}) = 0\vec{x} = \vec{0}$$

If *I* is the n x n identity matrix, then for every vector **x** in $\Re^n$, we will have an identity operator on $\Re^n$, T_I: $\Re^n \to \Re^n$, where T_I is called multiplication by *I*.

$$T_I(\vec{x}) = I\vec{x} = \vec{x}$$

Next, we will look at some important operators on $\Re^2$ and $\Re^3$, namely the linear operators that produce reflections, projections, and rotations.

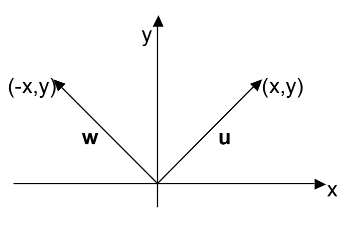
Linear Operators for Reflection

If the linear operator T: $\Re^2 \to \Re^2$ maps each vector into its symmetric image about the y-axis, we can construct a reflection operator or linear transformation as follows:

$$T(\vec{u}) = \vec{w} = (w_1, w_2) = (-x + 0y, 0x + y)$$

$$w_1 = -x + 0y \Rightarrow w_1 = -x$$

$$w_2 = 0x + y \Rightarrow w_2 = y$$



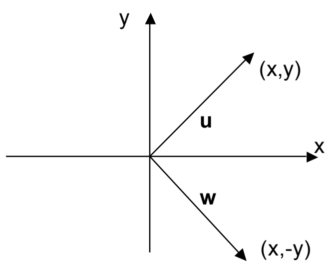
$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear Operators for Reflection (Cont.)

If the linear operator T: $\Re^2 \rightarrow \Re^2$ maps each vector into its symmetric image about the x-axis, we can construct a reflection operator or linear transformation as follows:

$$T(\vec{u}) = \vec{w} = (w_1, w_2) = (x + 0y, 0x - y)$$



$$w_1 = x + 0y \Rightarrow w_1 = x$$

$$w_2 = 0x - y \Rightarrow w_2 = -y$$

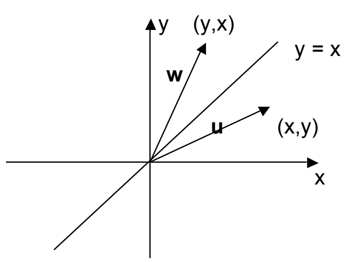
$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Linear Operators for Reflection (Cont.)

If the linear operator T: $\Re^2 \rightarrow \Re^2$ maps each vector into its symmetric image about the line y = x, we can construct a reflection operator or linear transformation as follows:

$$T(\vec{u}) = \vec{w} = (w_1, w_2) = (0x + y, x + 0y)$$



$$w_1 = 0x + y \Rightarrow w_1 = y$$

$$w_2 = x + 0y \Rightarrow w_2 = x$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Linear Operators for Reflection (Cont.)

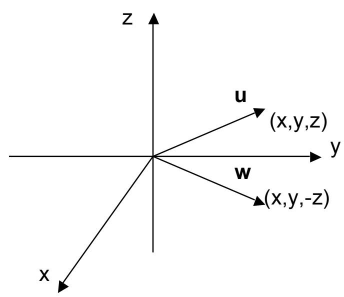
If the linear operator T: $\Re^3 \rightarrow \Re^3$ maps each vector into its symmetric image about the xy-plane, we can construct a reflection operator or linear transformation as follows:

$$T(\vec{u}) = \vec{w} = (w_1, w_2, w_3) = (x + 0y + 0z, 0x + y + 0z, 0x + 0y - z)$$

$$w_1 = x + 0y + 0z \Rightarrow w_1 = x$$

$$w_2 = 0x + y + 0z \Rightarrow w_2 = y$$

$$w_3 = 0x + 0y - z \Rightarrow w_3 = -z$$



$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Orthogonal Projection Operator

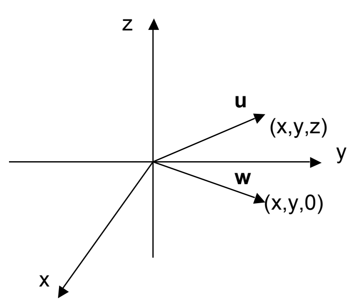
If the linear operator T: $\Re^3 \rightarrow \Re^3$ maps each vector into its orthogonal projection on the xy-plane, we can construct a projection operator or linear transformation as follows:

$$T(\vec{u}) = \vec{w} = (w_1, w_2, w_3) = (x + 0y + 0z, 0x + y + 0z, 0x + 0y + 0z)$$

$$w_1 = x + 0y + 0z \Rightarrow w_1 = x$$

$$w_2 = 0x + y + 0z \Rightarrow w_2 = y$$

$$w_3 = 0x + 0y + 0z \Rightarrow w_3 = 0$$



$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Orthogonal Projection Operator (Cont.)

Example: Use matrix multiplication to find the orthogonal projection of (-9,4,3) on the xy-plane.

From previous slide, the standard matrix for the linear operator T mapping each vector into its orthogonal projection on the xy-plane in $\Re^3$ is obtained:

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the orthogonal projection, **w**, of (-9,4,3) on the xy-plane is:

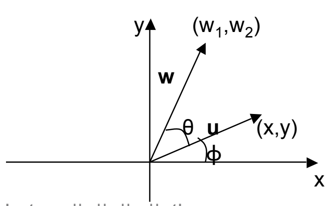
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -9 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \\ 0 \end{bmatrix}$$

Thus, T(-9,4,3) = (-9,4,0).

Linear Operators for Rotation

If the linear operator T: $\Re^2 \rightarrow \Re^2$ rotates each vector counterclockwise in $\Re^2$ through a fixed angle θ in $\Re^2$, we can construct a rotation operator or linear transformation as follows:

$$T(\vec{u}) = \vec{w} = (w_1, w_2) = (r\cos(\theta+\phi), r\sin(\theta+\phi))$$
$$= (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$$

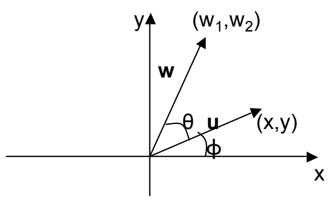


$$w_1 = r\cos(\theta+\phi)$$
$$= r\cos(\theta)\cos(\phi) - r\sin(\theta)\sin(\phi)$$
$$= x\cos(\theta) - y\sin(\theta)$$
$$w_2 = r\sin(\theta+\phi)$$
$$= r\sin(\theta)\cos(\phi) + r\cos(\theta)\sin(\phi)$$
$$= x\sin(\theta) + y\cos(\theta)$$

Hint: Let r = ||**u**||=||**w**||, then use x = r cos(ϕ), y = r sin(ϕ), w_1=r cos(θ+ϕ), w_2= r sin(θ+ϕ), and trigonometry identities.

Linear Operators for Rotation (Cont.)

$$w_1 = r\cos(\theta+\phi) = x\cos(\theta) - y\sin(\theta)$$
$$w_2 = r\sin(\theta+\phi) = x\sin(\theta) + y\cos(\theta)$$



$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Linear Operators for Rotation (Cont.)

Example: Use matrix multiplication to find the image of the vector (3,-4) when it is rotated through an angle, θ, of 30°.

Since the standard matrix for the linear operator T rotating each vector through an angle of θ (counterclockwise) in $\Re^2$ has been obtained:

$$[T] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Linear Operators for Rotation (Cont.)

It follows that the image, **w**, of (3,-4) when it is rotated through an angle of 30° (counterclockwise) in $\Re^2$ can be found as:

$$\vec{w} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3(\frac{\sqrt{3}}{2}) - 4(-\frac{1}{2}) \\ 3(\frac{1}{2}) - 4(\frac{\sqrt{3}}{2}) \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}+4}{2} \\ \frac{3-4\sqrt{3}}{2} \end{bmatrix}$$

Thus,

$$T(3,-4) = \left(\frac{3\sqrt{3}+4}{2}, \frac{3-4\sqrt{3}}{2} \right)$$

Composition of Linear Transformations

If T_A: $\Re^n \to \Re^k$ and T_B: $\Re^k \to \Re^m$ are linear transformations, then the application of T_A followed by T_B produces a transformation from $\Re^n$ to $\Re^m$; this transformation is called the composition of T_B with T_A and is denoted by $T_B \circ T_A$.

The composition $T_B \circ T_A$ is linear because

$$(T_B \circ T_A)(\vec{x}) = T_B(T_A(\vec{x}))$$
$$= B(A(\vec{x})) = (BA)(\vec{x})$$

Thus, $T_B \circ T_A$ is multiplication by BA and can be expressed as $T_B \circ T_A = T_{BA}$.

Alternatively, we have

$[T_B \circ T_A] = [T_B][T_A]$.

Composition of Linear Transformations (Cont.)

Example: Find the standard matrix for the stated composition of linear operators on $\Re^2$, if a rotation of π/2 is followed by a reflection about the line y = x.

We know the standard matrix for the linear operator T_A rotating each vector through an angle of θ = π/2 (counterclockwise) in $\Re^2$ is as follows:

$$[T_A] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Composition of Linear Transformations (Cont.)

We also know the standard matrix for the linear operator, T_B, reflecting each vector about the line y = x in $\Re^2$ is as follows:

$$[T_B] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The composition we want is the linear operator T:

$T = T_B \circ T_A$ (rotation followed by reflection).

Therefore, the standard matrix for T is

$[T] = [T_B \circ T_A] = [T_B][T_A]$.

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note: This is the symmetric image about the x-axis matrix. See slide 17.

What have we learned?

We have learned to:

1. Use vector notation in $\Re^n$.
2. Find the inner product of two vectors in $\Re^n$.
3. Find the norm of a vector and the distance between two vectors in $\Re^n$.
4. Express a linear system in $\Re^n$ in dot product form.
5. Find the standard matrix of a linear transformation from $\Re^n$ to $\Re^m$.
6. Use linear transformations such as reflections, projections, and rotations.
7. Use the composition of two or more linear transformations .

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

• Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition