MAC 2103

Module 7

Euclidean Vector Spaces II

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Learning Objectives

Upon completing this module, you should be able to:

1. Determine if a linear operator in $\Re^n$ is one-to-one.
2. Find the inverse of a linear operator in $\Re^n$.
3. Use the images of the standard basis vectors to find a standard matrix in $\Re^n$.
4. Find the polynomial q=T(p) in P_1 corresponding to the transformation T on any polynomials in P_1.

Euclidean Vector Spaces II

There are two major topics in this module:

Properties of Linear Transformations from $\Re^n$ to $\Re^m$

Linear Transformations and Polynomials

What are the Important Properties of Linear Transformations?

A transformation T: $\Re^n \rightarrow \Re^m$ is linear if both of the following relationships hold for all vectors **u** and **v** in $\Re^n$ and for every scalar *s*: (See Theorem 4.3.2)

a) T(**u** + **v**) = T(**u**) + T(**v**)

b) T(*s***u**) = *s*T(**u**)

What are the Important Properties of Linear Transformations? (Cont.)

It follows that:

T(-**v**) = T[(-1)**v**] = (-1)T(**v**) = - T(v),

T(**u** - **v**) = T[**u** + (-1) **v**] = T(**u**) + (-1)T(**v**) = T(**u**) - T(**v**),

T(**0**) = T(0**v**) = 0T(**v**) = **0**, since 0**v** = **0**;

and

$$T(\vec{v}) = T(s_1\vec{v}_1 + s_2\vec{v}_2 + \ldots + s_n\vec{v}_n)$$
$$= s_1T(\vec{v}_1) + s_2T(\vec{v}_2) + \ldots + s_nT(\vec{v}_n), \vec{v} = s_1\vec{v}_1 + s_2\vec{v}_2 + \ldots + s_n\vec{v}_n$$

How to Determine if a Linear Operator in $\Re^n$ is one-to-one?

Example: Find the standard matrix for the linear operator defined by the equations and determine whether the operator is one-to-one?

(a) $w_1 = 6x_1 - 3x_2$

$w_2 = 2x_1 - x_2$

Solution:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = [T] = \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}, \det(A) = 0$$

Since det(A) = 0, the matrix is not invertible. Thus, the linear operator in this case is not one-to-one.

How to Determine if a Linear Operator in $\Re^n$ is one-to-one? (Cont.)

Example: Find the standard matrix for the linear operator defined by the equations and determine whether the operator is one-to-one?

(b)
$$w_1 = 4x_1 - 3x_2$$
$$w_2 = 2x_1 + x_2$$

Solution:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = [T] = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}, \det(A) = 10 \neq 0$$

Since det(A) ≠ 0, the matrix is invertible. Thus, the linear operator in this case is one-to-one.

How to Find the Inverse of the Linear Operator in $\Re^n$?

Example: Find the inverse of the operator if the operator is one-to-one?

$$w_1 = 4x_1 - 3x_2$$
$$w_2 = 2x_1 + x_2$$

Solution:

From the previous slide, we have checked that the linear operator is one-to-one.

$$[T] = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}, \det(A) = 10$$

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

How to Find the Inverse of the Linear Operator in $\Re^n$? (Cont.)

Thus,

$$T^{-1}(w_1,w_2)=[T^{-1}]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}=\begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{10}w_1+\frac{3}{10}w_2 \\ -\frac{1}{5}w_1+\frac{2}{5}w_2 \end{bmatrix}$$

$$T^{-1}(w_1,w_2)=(\tfrac{1}{10}w_1+\tfrac{3}{10}w_2,-\tfrac{1}{5}w_1+\tfrac{2}{5}w_2)$$ **Check:** $T^{-1}T=I_2$

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What are the Standard Basis Vectors in $\Re^n$?

The standard basis vectors in $\Re^n$ are the columns of I_n (the identity matrix in $\Re^n$). We have represented the standard basis vectors in $\Re^3$ as **i**, **j** , and **k**; In order to extend the notations to $\Re^n$, we can represent them as $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ (note that the hat notation used below is generally reserved to denote unit vectors) as follows:

$$\hat{e}_1=\hat{i}=\vec{i}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\hat{e}_2=\hat{j}=\vec{j}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\hat{e}_3=\hat{k}=\vec{k}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}$$

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What are Standard Basis Vectors in $\Re^n$? (Cont.)

As mentioned previously, the standard basis vectors in $\Re^n$ are the columns of the I_n, we can represent them in $\Re^n$, as $\mathbf{e_1}$, $\mathbf{e_2}$, … , $\mathbf{e_n}$ as follows:

$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ \ldots, \ \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & . & . & . & \hat{e}_n \end{bmatrix}$$

Thus, $AI_n = \begin{bmatrix} A\hat{e}_1 & A\hat{e}_2 & . & . & . & A\hat{e}_n \end{bmatrix} = A$

How to Find the Standard Matrix from the Images of the Standard Basis Vectors in $\Re^n$?

Now we can use the images of the standard basis vectors to find the standard matrix.

As we have learned in module 6, if A is the standard matrix for T: $\Re^n \to \Re^m$, then A = $[T_A]$ = [T].

Thus,

$$I_n = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & . & . & . & \hat{e}_n \end{bmatrix}$$

$$AI_n = \begin{bmatrix} A\hat{e}_1 & A\hat{e}_2 & . & . & . & A\hat{e}_n \end{bmatrix} = A = [T_A] = [T].$$

$$[T] = \begin{bmatrix} T(\hat{e}_1) & T(\hat{e}_2) & . & . & . & T(\hat{e}_n) \end{bmatrix}$$

Note: If the linear transformation is represented by T: $\Re^n \to \Re^m$ or T_A: $\Re^n \to \Re^m$; the matrix $A = [a_{ij}]$ is called the standard matrix for the linear transformation, and T is called multiplication by A.

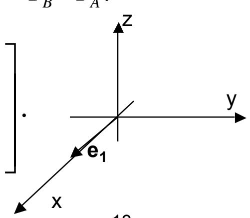
How to Find the Standard Matrix from the Images of the Standard Basis Vectors in ℜ^n? (Cont.)

Example: Find the standard matrix for T: ℜ³ → ℜ³ from the images of the standard basis vectors, where T: ℜ³ → ℜ³ reflects a vector about the xz-plane and then contracts that vector by a factor of 1/2.

Solution: We want to find the standard matrix from the images of the standard basis vectors,

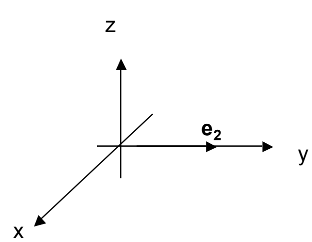
$$A = [T] = [T_B][T_A] = [\ T(\hat{e}_1)\quad T(\hat{e}_2)\quad T(\hat{e}_3)\ ], T = T_B \circ T_A.$$

$$T(\hat{e}_1) = T_B T_A(\hat{e}_1) = T_B T_A\begin{bmatrix}1\\0\\0\end{bmatrix}, T_A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$



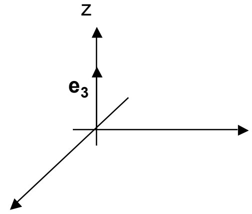
How to Find the Standard Matrix from the Images of the Standard Basis Vectors in ℜ^n? (Cont.)

$$T(\hat{e}_2) = T_B T_A(\hat{e}_2) = T_B T_A\begin{bmatrix}0\\1\\0\end{bmatrix}, T_A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix}.$$



How to Find the Standard Matrix from the Images of the Standard Basis Vectors in $\Re^n$? (Cont.)

$$T(\hat{e}_3)=T_BT_A(\hat{e}_3)=T_BT_A\begin{bmatrix}0\\0\\1\end{bmatrix},T_A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$



How to Find the Standard Matrix from the Images of the Standard Basis Vectors in $\Re^n$? (Cont.)

$$T(\hat{e}_1)=T_BT_A\begin{bmatrix}1\\0\\0\end{bmatrix}=T_B\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\0\\0\end{bmatrix}$$

$$T(\hat{e}_2)=T_BT_A\begin{bmatrix}0\\1\\0\end{bmatrix}=T_B\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-\frac{1}{2}\\0\end{bmatrix}$$

$$T(\hat{e}_3)=T_BT_A\begin{bmatrix}0\\0\\1\end{bmatrix}=T_B\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix}$$

The standard matrix is: $[T]=[\ T(\hat{e}_1)\quad T(\hat{e}_2)\quad T(\hat{e}_3)\ ]=\begin{bmatrix}\frac{1}{2}&0&0\\0&-\frac{1}{2}&0\\0&0&\frac{1}{2}\end{bmatrix}$

How to Find the Polynomial q=T(p) in P_1 Corresponding to the Transformation T on any Polynomials in P_1?

Example: What is the corresponding polynomial q=T(p) on polynomials of degree ≤ 1, P_1, for the multiplying matrix A:

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

Solution: First of all, let A be the multiplying matrix for the transformation T.

T is a linear operator on P_1 for which the domain is P_1 and the codomain is P_1.

How to Find the Polynomial q=T(p) in P_1 Corresponding to the Transformation T on any Polynomials in P_1?(Cont.)

Thus, if p is a polynomial of degree ≤ 1 and $p(x) = ax^1 + bx^0$ is a linear combination with real-valued coefficients of $x^1 = x$ and $x^0 = 1$, which are linearly independent functions (we will discuss this in module 8), for some real numbers a and b. Then, A multiplies the vector of coefficients of p(x).

$$\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a-2b \\ a+2b \end{bmatrix}$$

and the corresponding transformation on p(x) is as follows:

$$q(x) = (3a-2b)x + (a+2b)(1) = T(p) = q \in P_1$$

What have we learned?

We have learned to:

1. Determine if a linear operator in $\Re^n$ is one-to-one.
2. Find the inverse of a linear operator in $\Re^n$.
3. Use the images of the standard basis vectors to find a standard matrix in $\Re^n$.
4. Find the polynomial q=T(p) in P_1 corresponding to the transformation T on any polynomials in P_1.

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

• Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition