MAC 2103

Module 8

General Vector Spaces I

Learning Objectives

Upon completing this module, you should be able to:

1. Recognize from the standard examples of vector spaces, that a vector space is closed under vector addition and scalar multiplication.
2. Determine if a subset W of a vector space V is a subspace of V.
3. Find the linear combination of a finite set of vectors.
4. Find W = span(S), a subspace of V, given a set of vectors S in a vector space V.
5. Determine if a finite set of non-zero vectors in V is a linearly dependent set or linearly independent set.
6. Use the Wronskian to determine if a set of vectors that are differentiable functions is linearly independent.

General Vector Spaces I

There are three major topics in this module:

Real Vector Spaces or Linear Spaces
Subspaces
Linear Independence

What are the Standard Examples of Vector Spaces?

We have seen some of them before; some standard examples of vector spaces are as follows:

$$R^1, R^2, R^3, R^n, M_{m,n}, P_n, C(-\infty,\infty), C[a,b]$$

Can you identify them? We will look at some of them later in this module.

For now, know that we can always add any two vectors and multiply all vectors by a scalar within any vector space.

What are the Standard Examples of Vector Spaces? (Cont.)

Since we can always add any two vectors and multiply all vectors by a scalar in any vector space, we say that a vector space is closed under vector addition and scalar multiplication. In other words, it is closed under linear combinations.

A vector space is also called a linear space. In fact, a linear space is a better name.

What is a Vector Space?

Let V be a non-empty set of objects **u**, **v**, and **w**, on which two operations, vector addition and scalar multiplication, are defined. If V can satisfy the following ten axioms, then V is a vector space. (Please pay extra attention to axioms 1 and 6.)

1. If **u**, **v** $\in$ V, then **u** + **v** $\in$ V ~ Closure under addition
2. **u** + **v** = **v** + **u** ~ Commutative property
3. **u** + (**v** + **w**)= (**u** + **v**)+ **w** ~ Associative property
4. There is a unique zero vector such that **u** + **0** = **0** + **u** = **u**, for all **u** in V. ~ Additive identity
5. For each **u**, there is a unique vector -**u** such that **u** + (-**u**) = **0**. ~ Additive inverse

What is a Vector Space? (Cont.)

Here are the next five properties:

6. If k is in a field ($\Re$), k is a scalar and $\mathbf{u} \in V$, then $k\mathbf{u} \in V$ ~ Closure under scalar multiplication
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ ~ Distributive property
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$ ~ Distributive property
9. $k(m\mathbf{u}) = (km)\mathbf{u}$ ~ Associative property
10. $1\mathbf{u} = \mathbf{u}$ ~ Scalar identity

Looks familiar. You have used them in $\Re$, $\Re^2$, and $\Re^3$ before.

What is a Vector Space? (Cont.)

Example: Show that the set of all 4 x 3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

If A and B are 4 x 3 matrices and s is a scalar, then A + B and sA are also 4 x 3 matrices. Since the resulting matrices have the same form, the set is closed under matrix addition and matrix multiplication. We know from the previous modules that the other vector space axioms hold as well. Thus, we can conclude that the set is a vector space.

Similarly, we can show that the set of all m x n matrices, $M_{m,n}$, is a vector space.

What is a Subspace?

A subspace is a non-empty subset of a vector space; it is a subset that satisfies all the ten axioms of a vector space, including axioms 1 and 6:

- Closure under addition, and
- Closure under scalar multiplication.

How to Determine if a Subset W of a Vector Space V is Subspace of V?

Since a subset inherits the ten axioms from its larger vector space, to determine if a subset W of a vector space V is a subspace of V, we only need to check the following two axioms:

1. If **u** , **v** $\in$ W, then **u** + **v** $\in$ W ~ Closure under addition
2. If *k* is a scalar and **u** $\in$ W, then *k***u** $\in$ W ~ Closure under scalar multiplication

Note that the zero subspace = {**0**} and V itself are both valid subspaces of V. One is the smallest subspace of V, and one is the largest subspace of V.

How to Determine if a Subset W of a Vector Space V is Subspace of V? (Cont.)

Example: Is the following set of vectors a subspace of $\Re^3$?

u = (3, -2, 0) and **v** = (4, 5, 0).

Since a subset inherits the ten axioms from its larger vector space, to determine if a subset W of a vector space V is a subspace of V, we only need to verify the following two axioms:

1. If **u** , **v** $\in$ W, then **u** + **v** $\in$ W .
2. If k is any scalar and **u** $\in$ W, then k**u** $\in$ W.

Check:

u + **v** = (3+4, -2+5, 0+0) = (7, 3, 0) $\in$ W .

k**u** = (3k, -2k, 0) $\in$ W .

Thus, W is a subspace of $\Re^3$ and is the xy-plane in $\Re^3$.

What is a Linear Combination of Vectors?

By definition, a vector **w** is called a linear combination of the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, …, $\mathbf{v_r}$ if it can be expressed in the form

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_r\vec{v}_r = \sum_{i=1}^{r} k_i\vec{v}_i$$

where $k_1, k_2, \ldots, k_r$ are scalars.

For example, if we have a set of vectors in $\Re^3$, S = {$\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ }, where $\mathbf{v_1}$ = (2, 4, 3), $\mathbf{v_2}$ = (-1, 3, 1), and $\mathbf{v_3}$ = (8, 23, 17),

we can see that $\mathbf{v_3}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, since

$\mathbf{v_3} = 5\mathbf{v_1} + 2\mathbf{v_2}$ = 5(2, 4, 3) + 2(-1, 3, 1) = (8, 23, 17).

How to Find a Linear Combination of a Finite Set of Vectors?

Example: Let S = {**u**, **v**, **w**} ⊆ $\Re^3$=V. Express **p** = (-3,8,4) as linear combination of **u** = (1,1,2), **v** = (-1,3,0), and w = (0,1,2).

$$\vec{p} = k_1\vec{u}_1 + k_2\vec{v} + k_3\vec{w}$$

$$(-3,8,4) = k_1(1,1,2) + k_2(-1,3,0) + k_3(0,1,2)$$

$$(-3,8,4) = (k_1 - k_2, k_1 + 3k_2 + k_3, 2k_1 + 2k_3)$$

In order to solve for the scalars k_1, k_2, and k_3, we equate the corresponding components and obtain the system as follows:

$$A\vec{k} = \vec{p} \qquad \begin{aligned} k_1 - k_2 &= -3 \\ k_1 + 3k_2 + k_3 &= 8 \\ 2k_1 + 2k_3 &= 4 \end{aligned}$$

Note: If **u**, **v**, and **w** are vectors in a vector space V , then the set W = span(S) of all linear combinations of **u**, **v**, and **w** is a subspace of V; **p = (-3, 8, 4)** is just one of the linear combinations in the set W = span(S).

How to Find a Linear Combination of a Finite Set of Vectors? (Cont.)

We can solve this system using Gauss-Jordan Elimination.

$$[A \mid \vec{p}] = \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 1 & 3 & 1 & 8 \\ 2 & 0 & 2 & 4 \end{bmatrix}$$

$$\begin{matrix} r1 \\ -r1 + r2 \to r2 \\ -2r1 + r3 \to r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 4 & 1 & 11 \\ 0 & 2 & 2 & 10 \end{bmatrix}$$

$$\begin{matrix} r1 \\ \frac{1}{4}r2 \to r2 \\ r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 2 & 2 & 10 \end{bmatrix}$$

$$\begin{matrix} r1 \\ r2 \\ -2r2 + r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & \frac{3}{2} & \frac{9}{2} \end{bmatrix}$$

$$\begin{matrix} r1 \\ r2 \\ \frac{2}{3}r3 \to r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

How to Find a Linear Combination of a Finite Set of Vectors? (Cont.)

$$\begin{matrix} r1 \\ r2 \\ \frac{2}{3}r3 \rightarrow r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} r1 \rightarrow r1 \\ -\frac{1}{4}r3 + r2 \rightarrow r2 \\ r3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} r2 + r1 \rightarrow r1 \\ r2 \\ r3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$k_3 = 3, k_2 = 2, k_1 = -1$

Thus, the system is consistent and **p** can be expressed as a linear combination of **u**, **v**, and w as follows:

p = -**u** + 2**v** + 3**w**

Note: If the system is inconsistent, we will not be able to express **p** as a linear combination of **u**, **v**, and **w**. Then, **p** is not a linear combination of **u**, **v**, and **w**.

What is the Spanning Set?

Let S = {$\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_r$ } be a set of vectors in a vector space V, then there exists a subspace W of V consisting of all linear combinations of the vectors in S.

W is called the space spanned by $\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_r$. Alternatively, we say that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_r$ span W.

Thus, W = span(S) = span {$\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_r$ } and the set S is the spanning set of the subspace W.

In short, if every vector in V can be expressed as a linear combinations of the vectors in S, then S is the spanning set of the vector space V.

How to Find the Space Spanned by a Set of Vectors?

In our previous example, S = {**u**, **v**, **w** } = {(1,1,2),(-1,3,0),(0,1,2)} is a set of vectors in the vector space $\Re^3$, and

$$\vec{p} = A\vec{k} \in W, \vec{k} = (k_1, k_2, k_3) = (-1,2,3)$$

Is $(x_1, x_2, x_3) = \vec{x} \in W$? Or can we solve $\vec{x} = A\vec{k}$ for any **x**?

Yes, if A^{-1} exists. Find det(A) to see if there is a unique solution?

If we let W be the subspace of $\Re^3$ consisting of all linear combinations of the vectors in S, then **x** $\in$ W for any **x** $\in \Re^3$.

Thus, W = span(S) = $\Re^3$.

How to Determine if a Finite Set of Non-Zero Vectors is a Linearly Dependent Set or Linearly Independent Set?

Let S = {$\mathbf{v_1}$, $\mathbf{v_2}$,…, $\mathbf{v_r}$ } be a set of finite non-zero vectors in a vector space V. The vector equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + ... + k_r\vec{v}_r = \vec{0}$$

has at least one solution, namely the trivial solution , 0 = k_1= k_2= … = k_r. If the only solution is the trivial solution, then S is a linearly independent set. Otherwise, S is a linearly dependent set. If $\mathbf{v_1}$, $\mathbf{v_2}$,…, $\mathbf{v_r} \in \Re^n$, then the vector equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + ... + k_r\vec{v}_r = \vec{0} = A\vec{k}, A = [\vec{v}_1 \, \vec{v}_2 \, ... \, \vec{v}_r]$$

How to Use the Wronskian to Determine if a Set of Vectors that are Differentiable Functions is Linearly Independent?

Let S = { f_1, f_2, …, f_n } be a set of vectors in $C^{(n-1)}(-\infty,\infty)$. The Wronskian is

$$w(x) = \begin{vmatrix} f_1(x) & f_2(x) & . & . & . & f_n(x) \\ f_1^{'}(x) & f_2^{'}(x) & . & . & . & f_1^{'}(x) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & . & . & . & f_n^{(n-1)}(x) \end{vmatrix}$$

If the functions f_1, f_2, …, f_n have n-1 continuous derivatives on the interval $(-\infty,\infty)$, and if $w(x) \neq 0$ on the interval $(-\infty,\infty)$, then we can say that S is a linearly independent set of vectors in $C^{(n-1)}(-\infty,\infty)$.

How to Use the Wronskian to Determine if a Set of Vectors that are Differentiable Functions is Linearly Independent? (Cont.)

Example: Let S = { f_1, f_2, f_3 } = {5, e^{2x}, e^{3x} }. Show that S is a linearly independent set of vectors in $C^2(-\infty,\infty)$.

The Wronskian is

$$w(x) = \begin{vmatrix} 5 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ 0 & 4e^{2x} & 9e^{3x} \end{vmatrix} = 5\begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 5(18e^{2x}e^{3x} - 12e^{2x}e^{3x}) = 5(6e^{2x}e^{3x}) = 30e^{5x} \neq 0$$

Since $w(x) \neq 0$ on the interval $(-\infty,\infty)$, we can say that S is a linearly independent set of vectors in $C^2(-\infty,\infty)$, the linear space of twice continuously differentiable functions on $(-\infty,\infty)$.

What have we learned?

We have learned to:

1. Recognize from the standard examples of vector spaces, that a vector space is closed under vector addition and scalar multiplication.
2. Determine if a subset W of a vector space V is a subspace of V.
3. Find the linear combination of a finite set of vectors.
4. Find W = span(S), a subspace of V, given a set of vectors S in a vector space V.
5. Determine if a finite set of non-zero vectors in V is a linearly dependent set or linearly independent set.
6. Use the Wronskian to determine if a set of vectors that are differentiable functions is linearly independent.

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

- Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition