MAC 2103

Module 9

General Vector Spaces II

Learning Objectives

Upon completing this module, you should be able to:

1. Find the coordinate vector with respect to the standard basis for any vector in $\Re^n$.
2. Find the coordinate vector with respect to another basis.
3. Determine the dimension of a vector space V from a basis for V.
4. Find a basis for and the dimension of the null space of A, null(A).
5. Find a basis for and the dimension of the column space of A, col(A).
   a) Show that the non-leading columns of A are linearly dependent since they can be written as a linear combination of the leading columns of A.
   b) Show that the leading columns of A are linearly independent and therefore form a basis for col(A).
6. Find a basis for and the dimension of the row space of A, row(A).

General Vector Spaces II

There are three major topics in this module:

Coordinate Vectors

Basis and Dimension

Null Space, Column Space, and Row Space of a Matrix

Quick Review

In module 8, we have learned that if we let S = {$\mathbf{v}_1$, $\mathbf{v}_2$, … , $\mathbf{v}_r$} be a finite set of non-zero vectors in a vector space V, the vector equation

$$k_1\vec{v}_1 + k_2\vec{v}_2 + ... + k_r\vec{v}_r = \sum_{i=1}^{r} k_i\vec{v}_i = \vec{0}$$

has at least one solution, namely the trivial solution ,

$$0 = k_1 = k_2 = \dots = k_r.$$

If the only solution is the trivial solution, then S is a linearly independent set. Otherwise, S is a linearly dependent set.

If every vector in the vector space V can be expressed as a linear combination of the vectors in S, then S is the spanning set of the vector space V. If S is a linearly independent set, then S is a basis for V and span(S) = V.

What is the Coordinate Vector with Respect to the Standard Basis for any Vector?

The set of standard basis vectors in $\Re^n$ is B = {$\mathbf{e_1}$, $\mathbf{e_2}$, … , $\mathbf{e_n}$}. If $\mathbf{v} \in \Re^n$, then

$$\vec{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T = v_1\hat{e}_1 + v_2\hat{e}_2 + \dots + v_n\hat{e}_n$$

and has components $v_1, v_2, \dots, v_n$.

The coordinate vector $\mathbf{v_B}$ has the coefficients from the linear combination of the basis vectors as its components. So,

$$\vec{v}_B = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T.$$

A better name might be coefficient vector, but it is not used. So, **v** is its own coordinate vector with respect to the standard basis, $\vec{v} = \vec{v}_B$.

What is the Coordinate Vector with Respect to Another Basis?

Example: Let B_1 = {$\mathbf{v_1}$, $\mathbf{v_2}$}, with

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

To show B_1 is a basis, we solve

$$\sum_{i=1}^{2} c_i\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 = [\ \vec{v}_1 \quad \vec{v}_2\ ]\vec{c} = A\vec{c} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}.$$

If the only solution is

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

then $\mathbf{v_1}$, $\mathbf{v_2}$ are linearly independent, and B_1 = {$\mathbf{v_1}$, $\mathbf{v_2}$} is a linearly independent set and a basis for $\Re^2$.

What is the Coordinate Vector with Respect to Another Basis? (Cont.)

The $\det(A) = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1 \neq 0$

and A^{-1} exists, so the only solution to A**c** = **0** is

$A^{-1}A\mathbf{c} = I_2\mathbf{c} = A^{-1}\,\mathbf{0} = \mathbf{0}$; so, **c** = **0**.

Thus, $B_1 = \{\mathbf{v_1}, \mathbf{v_2}\}$, is a basis for $\Re^2$ and not the standard basis.

What is the Coordinate Vector with Respect to Another Basis? (Cont.)

Let $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = [\ \vec{v}_1 \quad \vec{v}_2\ ]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$= c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A\vec{c} = \vec{v}$$

We want to solve A**c** = **v** for **c**, the vector of coefficients, which is the coordinate vector, $\vec{v}_{B_1}$. We solve [A|**v**] as follows:

$$[A \mid \vec{v}] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix} \Leftrightarrow \begin{matrix} r1 \\ r2 - 2r1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Thus, $c_2 = 1, c_1 = 2$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

What is the Coordinate Vector with Respect to Another Basis? (Cont.)

So, the c_i that are the components of the coordinate vector $\vec{v}_{B_1}$ are the coefficients in the linear combination of the basis vectors for **v**.

Thus, the coordinate vector with respect to B_1 is

$$\vec{v}_{B_1} = (\vec{v})_{B_1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (c_1, c_2) = (2,1).$$

What is a Basis for a Vector Space, and What is the Dimension of a Vector Space?

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. Then $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for the vector space V if both of the following conditions hold:

1. S is a set of linearly independent vectors or it is a linearly independent set, and
2. The vectors in S can span the vector space V. This means that the

   span(S) = {all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$} = V.

The dimension of a vector space is the number of vectors in any basis for the vector space. dim(V) = n. The vector space V could have infinitely many bases, for V ≠ span({**0**}).

How to Find a Basis for and the Dimension of the Null Space of A, Null(A)?

Example: Find a basis for and the dimension of the null space of A, null(A), which is the solution space of the homogeneous system:

$$A\vec{x}=\vec{0} \Leftrightarrow \begin{array}{l} 2x_1+2x_2-x_3 \quad +x_5=0 \\ -x_1-x_2+2x_3-3x_4+x_5=0 \\ x_1+x_2-2x_3 \quad -x_5=0 \\ x_3+x_4+x_5=0. \end{array}$$

We shall use Gauss Elimination to obtain a row echelon form.

$$\begin{matrix} r1 \\ r2 \\ r3 \\ r4 \end{matrix}\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} = \left[A \mid \vec{0}\right] = \left[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4 \quad \vec{a}_5 \mid \vec{0}\right]$$

How to Find a Basis for and the Dimension of the Null Space of A, Null(A)?(Cont.)

1) $$\begin{matrix} \frac{1}{2}r1 \\ r2 \\ r3 \\ r4 \end{matrix}\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

2) $$\begin{matrix} r1 \\ r1+r2 \\ -r1+r3 \\ r4 \end{matrix}\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

3) $$\begin{matrix} r1 \\ \frac{2}{3}r2 \\ r3 \\ r4 \end{matrix}\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

4) $$\begin{matrix} r1 \\ r2 \\ \frac{3}{2}r2+r3 \\ -r2+r4 \end{matrix}\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}$$

5) $$\begin{matrix} r1 \\ r2 \\ -\frac{1}{3}r3 \\ r4 \end{matrix}\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}$$

6) $$\begin{matrix} r1 \\ r2 \\ r3 \\ -3r3+r4 \end{matrix}\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How to Find a Basis for and the Dimension of the Null Space of A, Null(A)?(Cont.)

$$\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[G \mid \vec{0} \right]$$

G has red leading 1's in column 1, column 3, and column 4. These are the three leading columns of G. G is a row-echelon form of A. These columns are linearly independent and correspond to the linearly independent columns of A, $\vec{a}_1, \vec{a}_3, \vec{a}_4$, which form a basis for the column space of A, col(A).

The non-leading columns of G are column 2 and column 5 which correspond to linearly dependent columns of A and give us the free variables, $x_2 = s$ and $x_5 = t$ in our solution of the homogeneous system, A**x** = **0**.

How to Find a Basis for and the Dimension of the Null Space of A, Null(A)?(Cont.)

$$\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[G \mid \vec{0} \right]$$

We know $x_2 = s$ and $x_5 = t$.

Use back-substitution to find the solution **x**.

$$x_5 = t, x_4 = 0, x_3 = 2x_4 - x_5 = -t,$$

$$x_2 = s, x_1 = -x_2 + \tfrac{1}{2}x_3 - \tfrac{1}{2}x_5 = -s - t$$

All of the components of **x** are in terms of s and t.

How to Find a Basis for and the Dimension of the Null Space of A, Null(A)?(Cont.)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = s\vec{v}_1 + t\vec{v}_2$$

The solution **x** of A**x**=**0** is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$. The solution space is the span({$\mathbf{v_1}$, $\mathbf{v_2}$}), the set {$\mathbf{v_1}$, $\mathbf{v_2}$} is linearly independent (since $\mathbf{v_1}$ is not a multiple of $\mathbf{v_2}$) and is a basis for the solution space of the homogeneous system or the null space of A, null(A). So, null(A) = span({$\mathbf{v_1}$, $\mathbf{v_2}$}). Null(A) is a subspace of $\Re^5$ and has a dimension of 2, dim(null(A))=2.

Thus, $\vec{x} = s\vec{v}_1 + t\vec{v}_2$ for some s and t iff **x** is in the null(A).

How to Find a Basis for and the Dimension of the Column Space of A, Col(A)?

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} A \mid \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 & \mid \vec{0} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} G \mid \vec{0} \end{bmatrix}$$

By definition, the column space of A, col(A) = span({$\mathbf{a_1}$,$\mathbf{a_2}$,$\mathbf{a_3}$ $\mathbf{a_4}$,$\mathbf{a_5}$}), but not all of the vectors in the set are linearly independent.

The linearly independent columns of A correspond to the leading columns of G; hence, col(A) = span({$\mathbf{a_1}$,$\mathbf{a_3}$,$\mathbf{a_4}$}), and {$\mathbf{a_1}$,$\mathbf{a_3}$,$\mathbf{a_4}$} is a basis for col(A). Then dim(col(A))=3. We will prove these statements for A in the next few slides.

How to Find a Basis for and the Dimension of the Column Space of A, Col(A)? (Cont.)

$$\left[G\,|\,\vec{0}\right]=\begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \frac{1}{2}r2+r1 \\ r2 \\ r3 \\ r4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} r3+r1 \\ 2r3+r2 \\ r3 \\ r4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix}\,|\,\vec{0}\right] = \left[J\,|\,\vec{0}\right]$$

The last matrix is the reduced row-echelon form for [A|**0**].

We can see that $\mathbf{v}_2=(1)\mathbf{v}_1+(0)\mathbf{v}_3+(0)\mathbf{v}_4=\mathbf{v}_1$ or $\mathbf{v}_2=\mathbf{v}_1$. $\mathbf{v}_5=(1)\mathbf{v}_1+(1)\mathbf{v}_3+(0)\mathbf{v}_4=\mathbf{v}_1+\mathbf{v}_3$ or $\mathbf{v}_5=\mathbf{v}_1+\mathbf{v}_3$. Thus, $\mathbf{v}_2$ and $\mathbf{v}_5$ are linearly dependent columns in the span of $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$.

How to Find a Basis for and the Dimension of the Column Space of A, Col(A)?(Cont.)

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} = \left[A\,|\,\vec{0}\right] = \left[\begin{matrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{matrix}\,|\,\vec{0}\right]$$

The linear dependencies will hold for the columns of A. We can see that

$\mathbf{a}_2=\mathbf{a}_1$, and $\mathbf{a}_5=\mathbf{a}_1+\mathbf{a}_3$.

Thus, $\mathbf{a}_2$ and $\mathbf{a}_5$ are linearly dependent columns in the span$\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ = col(A).

How to Find a Basis for and the Dimension of the Column Space of A, Col(A)? (Cont.)

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & | \vec{0} \end{bmatrix} = [J | \vec{0}]$$

J is the reduced echelon form of A. The leading columns $\mathbf{v}_1$, $\mathbf{v}_3$, $\mathbf{v}_4$ are the linearly independent columns of J, since they are standard basis vectors in $\Re^4$. $\mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{v}_3 = \mathbf{e}_2$, $\mathbf{v}_4 = \mathbf{e}_3$.

$$c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3 = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0$ which proves linear independence.

How to Find a Basis for and the Dimension of the Column Space of A, Col(A)? (Cont.)

Now, we will show that {$\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_4$} is a linearly independent set. Let $c_1\vec{a}_1 + c_2\vec{a}_3 + c_3\vec{a}_4 = \vec{0}$ and let E be the product of all elementary matrices, such that EA=J. Then,

$$EA = \begin{bmatrix} E\vec{a}_1 & E\vec{a}_2 & E\vec{a}_3 & E\vec{a}_4 & E\vec{a}_5 \end{bmatrix} = J = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix}$$

with leading columns E$\mathbf{a}_1$=$\mathbf{v}_1$=$\mathbf{e}_1$, E$\mathbf{a}_3$=$\mathbf{v}_3$=$\mathbf{e}_2$, E$\mathbf{a}_4$=$\mathbf{v}_4$=$\mathbf{e}_3$, as seen from the previous slide. Multiplying by E gives us

$$E(c_1\vec{a}_1 + c_2\vec{a}_3 + c_3\vec{a}_4) = c_1E\vec{a}_1 + c_2E\vec{a}_3 + c_3E\vec{a}_4$$
$$= c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3 = \vec{c} = \vec{0}.$$

c = **0** is the unique solution since E is invertible. Therefore, {$\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_4$} is a linearly independent set and a basis for the col(A). Then, dim(col(A)) = 3.

How to Find a Basis for and the Dimension of the Row Space of A, Row(A)?

$$[J]=\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vec{w}_3 \\ \vec{w}_4 \end{bmatrix}$$

The nonzero row vectors in the matrix J are linearly independent. The row vectors $\mathbf{w_1}$,$\mathbf{w_2}$,$\mathbf{w_3}$ form a basis for the row space of J. Likewise, the nonzero row vectors in the matrix G are linearly independent and those three row vectors form a basis for row(G).

How to Find a Basis for and the Dimension of the Row Space of A, Row(A)? (Cont.)

Elementary row operations are linear operators from $\Re^5$ into $\Re^5$ and the new rows are linear combinations of the original rows. So, row(A) = row(G) = row(J), and the span({$\mathbf{w_1}$, $\mathbf{w_2}$, $\mathbf{w_3}$ }) = row(A). Then dim(row(A))=3.

In A, 2$\mathbf{w_1}$-$\mathbf{w_2}$ = $\mathbf{r_1}$, -$\mathbf{w_1}$+2$\mathbf{w_2}$-3$\mathbf{w_3}$ = $\mathbf{r_2}$, $\mathbf{w_1}$-2$\mathbf{w_2}$ = $\mathbf{r_3}$, and $\mathbf{w_2}$+$\mathbf{w_3}$ = $\mathbf{r_4}$. This proves row(J) = row(A). A basis for the row(A) is {$\mathbf{w_1}$, $\mathbf{w_2}$, $\mathbf{w_3}$ } = {[1 1 0 0 1], [0 0 1 0 1], [0 0 0 1 0]}.

The row(A) = col(A^T) since the rows of A are the columns of A^T. If we only find a basis for row(A), then we can find a basis for col(A^T) and switch the column vectors back to row vectors (see Example 8 on page 274).

What have we learned?

We have learned to :

1. Find the coordinate vector with respect to the standard basis for any vector in $\Re^n$.
2. Find the coordinate vector with respect to another basis.
3. Determine the dimension of a vector space V from a basis for V.
4. Find a basis for and the dimension of the null space of A, null(A).
5. Find a basis for and the dimension of the column space of A, col(A).
   a) Show that the non-leading columns of A are linearly dependent since they can be written as a linear combination of the leading columns of A.
   b) Show that the leading columns of A are linearly independent and therefore form a basis for col(A).
6. Find a basis for and the dimension of the row space of A, row(A).

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

• Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition