MAC 1140

Module 10
System of Equations and Inequalities II

Learning Objectives

Upon completing this module, you should be able to

1. represent systems of linear equations with matrices.
2. transform a matrix into row-echelon form.
3. perform Gaussian elimination.
4. use matrix notation.
5. find sums, differences, and scalar multiples of matrices.
6. find matrix products.
7. find matrix inverses symbolically.

Learning Objectives (Cont.)

8. represent linear systems with matrix equations.
9. solve linear systems with matrix inverses.
10. define and calculate determinants.
11. use determinants to find areas of regions.
12. apply Cramer's rule.
13. state limitations on the method of cofactors and Cramer's rule.

System of Equations and Inequalities II

There are four sections in this module:

What Method Can Be Used to Solve Any System of Linear Equations?

We have learned how to solve a system with three variables by using elimination and substitution.

In this module, we are going to learn a numerical method, which will combine these two methods, to solve any system of linear equations that could contain thousands of variables.

This numerical method is called Gaussian elimination. To make use of this method, we need to know: Matrix.

What is a Matrix?

A **matrix** is a rectangular array of elements.

$$\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 4 \\ -2 & -3 & 8 \\ 6 & 8 & -7 \end{bmatrix} \quad \begin{bmatrix} 4 & -8 & 3 \\ 2 & 4 & 7 \end{bmatrix}$$

The dimension of a matrix is *m* x *n*, if it has *m* rows and *n* columns. A square matrix has the same number of rows and columns. The first two matrices above are square matrices, with the dimension of 2 x 2 and 3 x 3.

We use matrix to represent our system of linear equations. When the matrix includes the constants of the linear equations, it is called an augmented matrix.

What is the Dimension of an Augmented Matrix?

Express the linear system with an augmented matrix. State the dimensions of the matrix.

$$4x + 2y = 5$$
$$-3x - 4y = 12$$

Solution

The system has two equations with two variables.

The augmented matrix has dimensions 2 x 3.

$$\left[\begin{array}{cc|c} 4 & 2 & 5 \\ -3 & -4 & 12 \end{array}\right]$$

Note: 5 and 12 are the “constants” in our linear system above.

Example

Write the linear system represented by the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 3 & 1 & 4 & -2 \\ -1 & 2 & 3 & 5 \end{array}\right]$$

Solution

$$2x + \quad 3z = 4$$
$$3x + y + 4z = -2$$
$$-x + 2y + 3z = 5$$

x represents the first column
y represents the second column
z represents the third column
The vertical line corresponds to the location of the equals sign.
The last column represents the constant terms.

How to Perform a Gauss Elimination?

MATRIX ROW TRANSFORMATIONS

For any augmented matrix representing a system of linear equations, the following row transformations result in an equivalent system of linear equations.

1. Any two rows may be interchanged.
2. The elements of any row may be multiplied by a nonzero constant.
3. Any row may be changed by adding to (or subtracting from) its elements a multiple of the corresponding elements of another row.

We use Gaussian Elimination to transform an augmented matrix into row-echelon form.

A matrix is in reduced row-echelon form if every element above and below a leading 1 in a column is 0. Once an augmented matrix is in reduced row-echelon form, the system of linear equations is solved.

Example of Transforming a Matrix into Row-Echelon Form

Use Gaussian elimination and backward substitution to solve the linear system of equations.

$$\begin{aligned} x + y + z &= 3 \\ -x + y + z &= -1 \\ y - 2z &= -5 \end{aligned}$$

Solution

Write the augmented matrix.

We need a zero in the highlighted area.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & -2 & -5 \end{array}\right]$$

We can add rows 1 and 2 denoted $R_1 + R_2$. The row that is changing is written first.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & -2 & -5 \end{array}\right]$$

Example of Transforming a Matrix into Row-Echelon Form (Cont.)

We need a 1 in the highlighted area.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & -2 & -5 \end{array}\right]$$

$\frac{1}{2}R_2$ ® $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & -5 \end{array}\right]$

We need a zero in the highlighted area.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & -5 \end{array}\right]$$

$$R_3 - R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -6 \end{array}\right]$$

We need a 1 in the highlighted area.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -6 \end{array}\right]$$

$$-\frac{1}{3}R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example of Transforming a Matrix into Row-Echelon Form (Cont.)

Because there is a one in the highlighted box, the matrix is now in row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

z = 2, apply backward substitution to find x and y

$$\begin{aligned} y + z &= 1 \\ y + 2 &= 1 \\ y &= -1 \end{aligned} \qquad \begin{aligned} x + y + z &= 3 \\ x + (-1) + 2 &= 3 \\ x &= 2 \end{aligned}$$

The solution of the system is (2, −1, 2).

Example

Let's try to transform the matrix from the previous example into row-reduced echelon form without using backward substitution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Need a 0 in the highlighted area.

$$R_2 - R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example (cont.)

Need zeros in the highlighted areas.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The solution is (2, −1, 2).

One More Example

Use Gaussian elimination to solve the system of linear equations, if possible.

$$\begin{aligned} 3x + 5y - z &= -2 \\ 4x - y + 2z &= 1 \\ -6x - 10y + 2z &= 0 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 5 & -1 & -2 \\ 4 & -1 & 2 & 1 \\ -6 & -10 & 2 & 0 \end{array}\right]$$

$$R_3 + 2R_1 \text{ ® } \left[\begin{array}{ccc|c} 3 & 5 & -1 & -2 \\ 4 & -1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

The last row is a false statement. $0 \neq 4$.

There are no solutions to the system of equations.

Matrix Notation

A general element of a matrix is denoted a_{ij}. The refers to an element in the *i*th row, *j*th column.

For example, $a_{3,1}$ would be an element in matrix A located in the third row, first column.

The matrices are equal if corresponding elements are equal.

Example

If $A = \begin{bmatrix} 6 & 5 & -2 \\ 3 & 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 & -3 \\ 5 & -2 & 4 \end{bmatrix}$, find

a) *A* + B b) *B* – *A*

Solution

a) $$A + B = \begin{bmatrix} 6 & 5 & -2 \\ 3 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 7 & 8 & -3 \\ 5 & -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 6+7 & 5+8 & -2+(-3) \\ 3+5 & 0+(-2) & -4+4 \end{bmatrix}$$
$$= \begin{bmatrix} 13 & 13 & -5 \\ 8 & -2 & 0 \end{bmatrix}$$

Example (cont.)

If $A = \begin{bmatrix} 6 & 5 & -2 \\ 3 & 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 & -3 \\ 5 & -2 & 4 \end{bmatrix}$, find

a) *A* + B b) *B* – *A*

Solution

b) $$B - A = \begin{bmatrix} 7 & 8 & -3 \\ 5 & -2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 5 & -2 \\ 3 & 0 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 7-6 & 8-5 & -3-(-2) \\ 5-3 & -2-0 & -4-(-4) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & -1 \\ 2 & -2 & 8 \end{bmatrix}$$

Some Operations on Matrices

OPERATIONS ON MATRICES

Matrix Addition

The sum of two $m \times n$ matrices A and B is the $m \times n$ matrix $A + B$, in which each element is the sum of the corresponding elements of A and B. This is written as $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$. If A and B have different dimensions, then $A + B$ is undefined.

Matrix Subtraction

The difference of two $m \times n$ matrices A and B is the $m \times n$ matrix $A - B$, in which each element is the difference of the corresponding elements of A and B. This is written as $A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$. If A and B have different dimensions, then $A - B$ is undefined.

Multiplication of a Matrix by a Scalar

The product of a scalar (real number) k and an $m \times n$ matrix A is the $m \times n$ matrix kA, in which each element is k times the corresponding element of A. This is written as $kA = k[a_{ij}] = [ka_{ij}]$.

Note: There is no division of matrices.

Example of Performing Matrix Addition and Matrix Subtraction

If possible, perform the indicated operations using

$$A = \begin{bmatrix} 5 & -3 \\ 4 & 6 \end{bmatrix},\ B = \begin{bmatrix} 0 & 2 \\ -3 & 4 \end{bmatrix},\ C = \begin{bmatrix} -2 & -3 & 6 \end{bmatrix}$$

a) $A + 2B$ b) $3A - 2B$

Solution

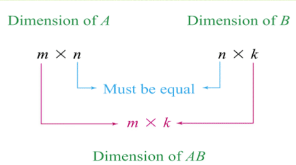
a) $$A + 2B = \begin{bmatrix} 5 & -3 \\ 4 & 6 \end{bmatrix} + 2\begin{bmatrix} 0 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -2 & 14 \end{bmatrix}$$

b) $$3A - 2B = 3\begin{bmatrix} 5 & -3 \\ 4 & 6 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -9 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 15 & -13 \\ 18 & 10 \end{bmatrix}$$

How to Perform Matrix Multiplication?

MATRIX MULTIPLICATION

The **product** of an $m \times n$ matrix A and an $n \times k$ matrix B is the $m \times k$ matrix AB, which is computed as follows. To find the element of AB in the ith row and jth column, multiply each element in the ith row of A by the corresponding element in the jth column of B. The sum of these products will give the element of row i, column j in AB.



Note:
The number of columns in the first matrix must equal to the number of rows in the second matrix.

Example of Multiplying Matrices

If possible, compute each product using

$$A=\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix},\ B=\begin{bmatrix}1 & 4\\ 2 & -3\\ -1 & 3\end{bmatrix},\ C=\begin{bmatrix}5 & 1\\ -1 & 2\end{bmatrix}$$

a) *AC* b) *BA*

Solution

a) The dimension of *A* is 2 x 3 and the dimension of *C* is 2 x 2. Therefore *AC* is undefined. The number of columns in *A* does not match the number of rows in *C*.

Example of Performing Matrix Multiplication (cont.)

b) *BA*

B	*A*		BA
3 x 2	2 x 3	=	3 x 3

$$BA\begin{bmatrix}1 & 4\\ 2 & -3\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & 1 & -1\\ 2 & 0 & 3\end{bmatrix}=\begin{bmatrix}1(3)+4(2) & 1(1)+4(0) & 1(-1)+4(3)\\ 2(3)+-3(2) & 2(1)+-3(0) & 2(-1)+-3(3)\\ -1(3)+3(2) & -1(1)+3(0) & -1(-1)+3(3)\end{bmatrix}$$

$$=\begin{bmatrix}11 & 1 & 11\\ 0 & 2 & -11\\ 3 & -1 & 10\end{bmatrix}$$

One More Example

If $A=\begin{bmatrix}2 & 0 & 1\\ 3 & -1 & 2\\ -4 & 1 & 3\end{bmatrix},\ B=\begin{bmatrix}4 & -5 & 6\\ 0 & 2 & 5\\ 3 & 1 & 2\end{bmatrix}$ find *AB*.

Solution $AB=\begin{bmatrix}2 & 0 & 1\\ 3 & -1 & 2\\ -4 & 1 & 3\end{bmatrix}\begin{bmatrix}4 & -5 & 6\\ 0 & 2 & 5\\ 3 & 1 & 2\end{bmatrix}$

$$=\begin{bmatrix}2(4)+0(0)+1(3) & 2(-5)+0(2)+1(1) & 2(6)+0(5)+1(2)\\ 3(4)+-1(0)+2(3) & 3(-5)+-1(2)+2(1) & 3(6)+-1(5)+2(2)\\ -4(4)+1(0)+3(3) & -4(-5)+1(2)+3(1) & -4(6)+1(5)+3(2)\end{bmatrix}$$

$$=\begin{bmatrix}11 & -9 & 14\\ 18 & -15 & 17\\ -7 & 25 & -13\end{bmatrix}$$

Basic Properties of Matrices

PROPERTIES OF MATRICES

Let A, B, and C be matrices. Assume that each matrix operation is defined.

1. $A + B = B + A$	Commutative property for matrix addition (No commutative property for matrix multiplication)
2. $(A + B) + C = A + (B + C)$	Associative property for matrix addition
3. $(AB)C = A(BC)$	Associative property for matrix multiplication
4. $A(B + C) = AB + BC$	Distributive property

What is Identity Matrix?

THE $n \times n$ IDENTITY MATRIX

The $n \times n$ **identity matrix**, denoted I_n, has only 1's on its main diagonal and 0's elsewhere.

The identity matrix is used to verified matrix inverses.

Inverses of a Square Matrix

INVERSE OF A SQUARE MATRIX

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix, denoted A^{-1}, that satisfies

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n,$$

then A^{-1} is the **inverse** of A.

Note that not every matrix has an inverse.

If the inverse of A exists, then A is invertible or nonsingular.
If a matrix A is not invertible, then it is singular.

Example of Verifying an Inverse

Determine if *B* is the inverse of *A*, where

$$A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Solution

For *B* to be the inverse of *A*, it must satisfy the equations $AB = I_2$ and $BA = I_2$.

$$AB = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

B is the inverse of *A*.

How to Represent the linear systems with matrix equations?

Represent the system of linear equations in the form *AX* = *B*.

$$\begin{aligned} y - z &= -4 \\ 4x + y &= -3 \\ 3x - y + 3z &= 1 \end{aligned}$$

Solution

The equivalent matrix equation is

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & 1 & 0 \\ 3 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix} = B$$

How to Find the Inverse Symbolically?

Find A^{-1} if $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

Solution Begin by forming a 2 · 4 augmented matrix. Perform row operations to obtain the identity matrix on the left side.

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] R_2 \leftrightarrow R_1 \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right] R_2 - 3R_1 \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$-1R_2 \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 \end{array}\right] R_1 - 2R_2 \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

Example

Write the linear system as a matrix equation in the form $AX = B$. Find A^{-1} and solve for X.

$$3x + 5y = 9$$
$$x + 2y = 4$$

Solution

$$AX = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} = B$$

This inverse was found in a previous example.

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \qquad X = A^{-1}B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The solution to the system is (−2, 3)

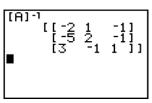
Another Example

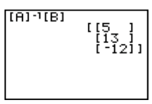
Write the linear system as a matrix equation of the form $AX = B$. Find A^{-1} and solve for X.

$$x + z = -7$$
$$2x + y + 3z = -13$$
$$-x + y + z = -4$$

Solution We can find the inverse by hand or with a graphing calculator.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ -13 \\ -4 \end{bmatrix} = B$$





What is the Determinant of a 2 x 2 Matrix?

A determinant is a real number associated with a square matrix.

DETERMINANT OF A 2 × 2 MATRIX

The **determinant** of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a real number defined by

$$\det A = ad - cb.$$

Determinants are commonly used to test if a matrix is invertible and to find the area of certain geometric figures.

How to Determine a Matrix is Invertible?

The following is often used to determine if a square matrix is invertible.

INVERTIBLE MATRIX

A square matrix A is invertible if and only if $\det A \neq 0$.

Example

Determine if A^{-1} exists by computing the determinant of the matrix A.

a) $A = \begin{bmatrix} -5 & 9 \\ 4 & -1 \end{bmatrix}$ b) $A = \begin{bmatrix} 9 & 3 \\ -3 & -1 \end{bmatrix}$

Solution

a) $\det A = \det \begin{bmatrix} -5 & 9 \\ 4 & -1 \end{bmatrix} = (-5)(-1) - (4)(9) = -31$

A^{-1} does exist

b) $\det B = \det \begin{bmatrix} 9 & 3 \\ -3 & -1 \end{bmatrix} = (9)(-1) - (-3)(3) = 0$

A^{-1} does not exist

What are Minors and Cofactors?

We know we can find the determinants of 2 x 2 matrices; but can we find the determinants of 3 x 3 matrices, 4 x 4 matrices, 5 x 5 matrices, ...?

In order to find the determinants of larger square matrices, we need to understand the concept of minors and cofactors.

MINORS AND COFACTORS

The **minor**, denoted by M_{ij}, for element a_{ij} in the square matrix A is the real number computed by performing the following steps.

STEP 1: Delete the ith row and jth column from the matrix A.

STEP 2: M_{ij} is equal to the determinant of the resulting matrix.

The **cofactor**, denoted A_{ij}, for a_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

Example of Finding Minors and Cofactors

Find the minor M_{11} and cofactor A_{11} for matrix A.

$$A = \begin{bmatrix} -8 & 0 & 4 \\ 4 & -6 & 7 \\ 2 & -3 & 5 \end{bmatrix}$$

Solution

To obtain M_{11} begin by crossing out the first row and column of A.

$$A = \begin{bmatrix} \cancel{-8} & \cancel{0} & \cancel{4} \\ \cancel{4} & -6 & 7 \\ \cancel{2} & -3 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} -6 & 7 \\ -3 & 5 \end{bmatrix}$$

The minor is equal to
$\det B = -6(5) - (-3)(7)$
$= -9$

Since $A_{11} = (-1)^{1+1}M_{11}$, A_{11} can be computed as follows:

$A_{11} = (-1)^2(-9) = -9$

How to Find the Determinant of Any Square Matrix?

Once we know how to obtain a cofactor, we can find the determinant of any square matrix. You may pick any row or column, but the calculation is easier if some elements in the selected row or column equal 0.

DETERMINANT OF A MATRIX USING THE METHOD OF COFACTORS

Multiply each element in any row or column of the matrix by its cofactor. The sum of the products is equal to the determinant.

Example

Find det A, if $A = \begin{bmatrix} -8 & 0 & 4 \\ 4 & -6 & 7 \\ 2 & -3 & 5 \end{bmatrix}$

Solution To find the determinant of A, we can select any row or column. If we begin expanding about the first column of A, then

$\det A = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

$A_{11} = -9$ from the previous example

$A_{21} = -12$

$A_{31} = 24$

$$\begin{aligned} \det A &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= (-8)(-9) + (4)(-12) + (2)(24) \\ &= 72 \end{aligned}$$

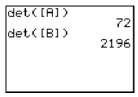
Example

Find the determinant of A and B using technology.

$$A = \begin{bmatrix} -8 & 0 & 4 \\ 4 & -6 & 7 \\ 2 & -3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -4 & 2 & 6 \\ 7 & 2 & -9 & 1 \\ 6 & 5 & 8 & 6 \\ -3 & 4 & 4 & 0 \end{bmatrix}$$

Solution

The determinant of A was calculated by hand in a previous example.



Example of Using Determinants to Find Area?

Determinants may be used to find the area of a triangle. If a triangle has vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3), then its area is equal to the absolute value of D, where

$$D = \frac{1}{2}\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{bmatrix}$$

If the vertices are entered into the columns of D in a counterclockwise direction, then D will be positive.

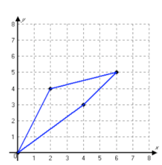
Example of Using Determinants to Find the Area of Parallelogram?

Find the area of the parallelogram.

Solution

View the parallelogram as two triangles.

The area is equal to the sum of the two triangles.



$$D = \frac{1}{2}\det\begin{bmatrix} 0 & 4 & 6 \\ 0 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix} = 1 \qquad D = \frac{1}{2}\det\begin{bmatrix} 0 & 6 & 2 \\ 0 & 5 & 4 \\ 1 & 1 & 1 \end{bmatrix} = 7$$

The area is 1 + 7 = 8.

What is Cramer's Rule?

CRAMER'S RULE FOR LINEAR SYSTEMS IN TWO VARIABLES

The solution to the linear system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

is given by $x = \frac{E}{D}$ and $y = \frac{F}{D}$, where

$$E = \det\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}, \quad F = \det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}, \quad \text{and} \quad D = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0.$$

Cramer's Rule is a method that utilizes determinants to solve systems of linear equations.

Example of Using Cramer's Rule to Solve the Linear System

Use Cramer's rule to solve the linear system.

$$x + 4y = 3$$
$$2x + 9y = 5$$

Solution In this system $a_1 = 1$, $b_1 = 4$, $c_1 = 3$, $a_2 = 2$, $b_2 = 9$ and $c_2 = 5$

$$E = \det\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 9 \end{bmatrix} = 27 - 20 = 7$$

$$F = \det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = 5 - 6 = -1$$

$$D = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = 9 - 8 = 1$$

Example of Using Cramer's Rule to Solve the Linear System (cont.)

$E = 7$, $F = -1$ and $D = 1$

The solution is

$$x = \frac{E}{D} = \frac{7}{1} = 7 \qquad y = \frac{F}{D} = \frac{-1}{1} = -1$$

Note that Gaussian elimination with backward substitution is usually more efficient than Cramer's Rule.

What Are the Limitations on the Method of Cofactors and Cramer's Rule?

The main limitations are as follow:

1. substantial number of arithmetic operations needed to compute determinants of large matrices.
2. the cofactor method of calculating the determinant of an n x n matrix, $n > 2$, generally involves more than $n!$ multiplication operations.
3. time and cost required to solve linear systems that involved thousands of equations in real-life applications.

What have we learned?

We have learned to

1. represent systems of linear equations with matrices.
2. transform a matrix into row-echelon form.
3. perform Gaussian elimination.
4. use matrix notation.
5. find sums, differences, and scalar multiples of matrices.
6. find matrix products.
7. find matrix inverses symbolically.

What have we learned? (Cont.)

8. represent linear systems with matrix equations.
9. solve linear systems with matrix inverses.
10. define and calculate determinants.
11. use determinants to find areas of regions.
12. apply Cramer's rule.
13. state limitations on the method of cofactors and Cramer's rule.

Credit

- Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:
- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition