MAC 1140

Module 12
Introduction to Sequences, Counting, The Binomial Theorem, and Mathematical Induction

Learning Objectives

Upon completing this module, you should be able to

1. represent sequences.
2. identify and use arithmetic sequences.
3. identify and use geometric sequences.
4. apply the fundamental counting principle.
5. calculate and apply permutations.
6. calculate and apply combinations.
7. derive the binomial theorem.

Learning Objectives (Cont.)

8. use the binomial theorem.
9. apply Pascal's triangle.
10. use mathematical induction to prove statements.
11. apply the generalized principle of mathematical induction.

Introduction to Sequences, Counting, The Binomial Theorem, and Mathematical Induction

There are four major topics in this module:

- Sequences
- Counting
- The Binomial Theorem
- Mathematical Induction

What is a Sequence?

- A sequence is a function that computes an ordered list.

Example

If an employee earns \$12 per hour, the function $f(n) = 12n$ generates the terms of the sequence

12, 24, 36, 48, 60, …

when $n = 1, 2, 3, 4, 5, \ldots$

What is an Infinite Sequence?

SEQUENCE

An **infinite sequence** is a function that has the set of natural numbers as its domain. A **finite sequence** is a function with domain $D = \{1, 2, 3, \ldots, n\}$, for some fixed natural number n.

What is the General Term of a Sequence?

- Instead of letting y represent the output, it is common to write $a_n = f(n)$, where n is a natural number in the domain of the sequence.
- The *terms* of a sequence are

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- The first term is $a_1 = f(1)$, the second term is $a_2 = f(2)$ and so on. The *nth term* or *general term* of a sequence is $a_n = f(n)$.

Example

Write the first four terms $a_1, a_2, a_3, a_4, \ldots$ of each sequence, where $a_n = f(n)$,

a) $f(n) = 5n + 3$ b) $f(n) = (4)^{n-1} + 2$

Solution

a) $a_1 = f(1) = 5(1) + 3 = 8$

$a_2 = f(2) = 5(2) + 3 = 13$

$a_3 = f(3) = 5(3) + 3 = 18$

$a_4 = f(4) = 5(4) + 3 = 23$

b) $a_1 = f(1) = (4)^{1-1} + 2 = 2$

$a_2 = f(2) = (4)^{2-1} + 2 = 6$

$a_3 = f(3) = (4)^{3-1} + 2 = 18$

$a_4 = f(4) = (4)^{4-1} + 2 = 66$

What is a Recursive Sequence?

- With a recursive sequence, one or more previous terms are used to generate the next term.
- The terms a_1 through a_{n-1} must be found before a_n can be found.

Example

a) Find the first four terms of the recursive sequence that is defined by

$a_n = 3a_{n-1} + 5$ and $a_1 = 4$, where $n \geq 2$.

b) Graph the first 4 terms of the sequence.

What is a Recursive Sequence? (cont.)

Solution

a) Numerical Representation

$$a_1 = 4$$
$$a_2 = 3a_1 + 5 = 3(4) + 5 = 17$$
$$a_3 = 3a_2 + 5 = 3(17) + 5 = 56$$
$$a_4 = 3a_3 + 5 = 3(56) + 5 = 173$$

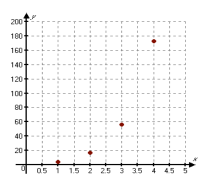
The first four terms are 4, 17, 56, and 173.

n	1	2	3	4
a_n	4	17	56	173

What is a Recursive Sequence? (cont.)

Graphical Representation

b) To represent these terms graphically, plot the points. Since the domain of the graph only contains natural numbers, the graph of the sequence is a scatterplot.



What is an Infinite Arithmetic Sequence?

INFINITE ARITHMETIC SEQUENCE

An **infinite arithmetic sequence** is a linear function whose domain is the set of natural numbers.

If the points of a sequence lie on a line, the sequence is arithmetic. In an arithmetic sequence, there is a common difference between adjacent points.

Example

An employee receives 10 vacations days per year. Thereafter the employee receives an additional 2 days per year with the company. The amount of vacation days after n years with the company is represented by

$f(n) = 2n + 10$, where f is a linear function.

How many vacation days does the employee have after 14 years? (Assume no rollover of days.)

Solution: $f(n) = 2n + 10$

$f(14) = 2(14) + 10 = 38$ days of vacation.

What is the Definition of an Arithmetic Sequence?

- An arithmetic sequence can be defined recursively by $a_n = a_{n-1} + d$, where d is a constant.
- Since $d = a_n - a_{n-1}$ for each valid n, d is called the *common difference*. If $d = 0$, then the sequence is a *constant sequence*. A ***finite arithmetic sequence*** is similar to an infinite arithmetic sequence except its domain is $D = \{1, 2, 3, \ldots, n\}$, where n is a fixed natural number.
- Since an arithmetic sequence is a linear function, it can always be represented by $f(n) = dn + c$, where d is the common difference and c is a constant.

Example

Find a general term $a_n = f(n)$ for the arithmetic sequence; $a_1 = 4$ and $d = -3$.

Solution

Let $f(n) = dn + c$.

Since $d = -3$, $f(n) = -3n + c$.

$a_1 = f(1) = -3(1) + c = 4$ or $c = 7$

Thus $a_n = -3n + 7$.

nth term of an Arithmetic Sequence

nth TERM OF AN ARITHMETIC SEQUENCE

In an arithmetic sequence with first term a_1 and common difference d, the nth term, a_n, is given by

$$a_n = a_1 + (n - 1)d.$$

Example

Find a symbolic representation (formula) for the arithmetic sequence given by 6, 10, 14, 18, 22,…

Solution

The first term is 6. Successive terms can be found by adding 4 to the previous term. $a_1 = 6$ and $d = 4$

$$a_n = a_1 + (n - 1)d$$
$$= 6 + (n - 1)(4)$$
$$= 4n + 2$$

What are Geometric Sequences?

- Geometric sequences are capable of either rapid growth or decay.

Examples

- Population
- Salary
- Automobile depreciation

What are Geometric Sequences? (cont.)

If the points of a sequence do not lie on a line, the sequence is not arithmetic. If each *y*-value after the first can be determined from the preceding one by multiplying by a common ratio, then this sequence is a geometric sequence.

INFINITE GEOMETRIC SEQUENCE

An **infinite geometric sequence** is a function defined by $f(n) = cr^{n-1}$, where c and r are nonzero constants. The domain of f is the set of natural numbers.

Example

Find a general term a_n for the geometric sequence; $a_3 = 18$ and $a_6 = 486$.

Solution

Find $a_n = cr^{n-1}$ so that $a_3 = 18$ and $a_6 = 486$.

Since $\frac{a_6}{a_3} = \frac{cr^{6-1}}{cr^{3-1}} = \frac{r^5}{r^2} = r^3$ and $\frac{a_6}{a_3} = \frac{486}{18} = 27,$

$r^3 = 27$ or $r = 3$.

So $a_n = c(3)^{n-1}$.

Therefore $a_3 = c(3)^{3-1} = 18$ or $c = 2$.

Thus $a_n = 2(3)^{n-1}$.

Fundamental Counting Principle

FUNDAMENTAL COUNTING PRINCIPLE

Let $E_1, E_2, E_3, \ldots, E_n$ be a sequence of n independent events. If event E_k can occur m_k ways for $k = 1, 2, 3, \ldots, n$, then there are

$$m_1 \cdot m_2 \cdot m_3 \cdot \cdots \cdot m_n$$

ways for all n events to occur.

Example

An exam contains five true-false questions and ten multiple-choice questions. Each multiple-choice question has four possible answers. Count the number of ways that the exam can be answered.

Solution

This is a sequence of 15 independent events. There are two ways to answer each of the first five questions.

There are four ways to answer the next 10 questions.

$$\underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ factors}} \times \underbrace{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}_{10 \text{ factors}}$$

$$2^5 \times 4^{10} = 33{,}554{,}432$$

What is a Permutation?

A permutation is an *ordering* or *arrangement*. For example, if three groups are scheduled to give a presentation in our class. The different arrangements of how these presentations can be taken place are called permutations. After the first group, there are two groups remaining for the second presentation. For the third presentation, there is only one possibility. The total number of permutations is equal to (3)(2)(1) = 6 or 3!

***n*-FACTORIAL**

For any natural number n,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

and

$$0! = 1.$$

Example

The values for

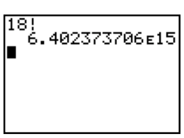
3! = (3)(2)(1) = 6

4! = (4)(3)(2)(1) = 24

5! = (5)(4)(3)(2)(1) = 120

a) Try to compute 7!.

b) Use a calculator to find 18!.



Solution

a) 7! = 7 · 6 · 5 · 4 · 3 · 2 · 1 = 5040

b) 18! =

Permutations of n Elements Taken r at a Time

PERMUTATIONS OF n ELEMENTS TAKEN r AT A TIME

If $P(n, r)$ denotes the number of permutations of n elements taken r at a time, with $r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!} = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}}.$$

Rev.S08

http://faculty.valenciacc.edu/ashaw/
Click link to download other modules.

Example

In how many ways can 4 students give a presentation in a class of 12 students.

Solution

The number of permutations of 12 elements taken 4 at a time.

$$P(12,4) = \underbrace{12 \times 11 \times 10 \times 9}_{4 \text{ factors}} = 11,880$$

Rev.S08

http://faculty.valenciacc.edu/ashaw/
Click link to download other modules.

What is the Difference Between Combination and Permutation?

A combination is not an ordering or arrangement, but rather a subset of a set of elements. Order is not important when finding combinations.

COMBINATIONS OF n ELEMENTS TAKEN r AT A TIME

If $C(n, r)$ denotes the number of combinations of n elements taken r at a time, with $r \leq n$, then

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!\, r!}.$$

Rev.S08

http://faculty.valenciacc.edu/ashaw/
Click link to download other modules.

Example

In how many ways can a committee of 3 people be chosen from a group of 10?

Solution

The order in which the committee is selected is not important.

$$C(10,3)=\frac{10!}{(10-3)!3!}=\frac{10!}{7!3!}=120$$

Rev.S08

http://faculty.valenciacc.edu/ashaw/
Click link to download other modules.

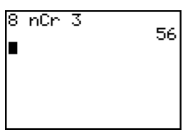
28

Example

Calculate $C(8, 3)$. Support your answer by using a calculator.

Solution

$$C(8,3)=\frac{8!}{(8-3)!3!}=\frac{8!}{5!3!}=56$$



Rev.S08

http://faculty.valenciacc.edu/ashaw/
Click link to download other modules.

29

Another Example

How many committees of 4 people can be selected from 7 women and 5 men, if a committee must consist of at least 2 men?

Solution

Two Men: Committee would be 2 men and 2 women.

$$C(5,2)\times C(7,2)=10\times 21=210$$

Three Men: Committee would be 3 men and 1 woman

$$C(5,3)\times C(7,1)=10\times 7=70$$

Four Men: Committee would be 4 men and 0 women

$$C(5,4)\times C(7,0)=5\times 1=5$$

The total number of committees would be

210 + 70 + 5 = 285

Rev.S08

http://faculty.valenciacc.edu/ashaw/
Click link to download other modules.

30

The Binomial Theorem

- Expanding expressions in the form $(a + b)^n$, where n is a natural number.
- Expressions occur in statistics, finite mathematics, computer science, and calculus.
- Combinations play a central role.

BINOMIAL THEOREM

For any positive integer n and numbers a and b,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}b^n.$$

The Binomial Theorem (cont.)

- Since $\binom{n}{r} = C(n,r)$ the combination formula

$C(n,r) = \dfrac{n!}{(n-r)!r!}$ can be used to evaluate

binomial coefficients.

Example

Use the binomial theorem to expand the expression $(3x + 1)^5$.

Solution

$$(3x+1)^5 = \binom{5}{0}(3x)^5 + \binom{5}{1}(3x)^4 1^1 + \binom{5}{2}(3x)^3 1^2 +$$

$$\binom{5}{3}(3x)^2 1^3 + \binom{5}{4}(3x)^1 1^4 + \binom{5}{5}1^5$$

$$= \frac{5!}{5!0!}(243x^5) + \frac{5!}{4!1!}(81x^4) + \frac{5!}{3!2!}(27x^3) +$$

$$\frac{5!}{2!3!}(9x^2) + \frac{5!}{1!4!}(3x) + \frac{5!}{0!5!}$$

$$= 243x^5 + 405x^4 + 270x^3 + 90x^2 + 15x + 1$$

Pascal's Triangle

- It can be used to efficiently compute the binomial coefficients $C(n,r)$.
- The triangle consists of ones along the sides.
- Each element inside the triangle is the sum of the two numbers above it.
- It can be extended to include as many rows as needed.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

Example

Expand $(2x - 5)^4$.

Solution

To expand $(2x - 5)^4$, let $a = 2x$ and $b = -5$ in the binomial theorem. We can use the fifth row of Pascal's triangle to obtain the coefficients 1, 4, 6, 4, and 1.

$$(2x-5)^4 = 1(2x)^4 + 4(2x)^3(-5)^1 + 6(2x)^2(-5)^2 + 4(2x)^1(-5)^3 + 1(-5)^4$$
$$= 16x^4 - 160x^3 + 600x^2 - 1000x + 625$$

How to Find the *k*th term?

- The binomial theorem gives all the terms of $(a + b)^n$.
- An individual term can be found by noting that the $(r + 1)$st term in the binomial expansion for $(a + b)^n$ is given by the formula

$$\binom{n}{r} a^{n-r} b^r, \text{ for } 0 \le r \le n.$$

Example of Finding the *k*th term

Find the fifth term of $(x + y)^{10}$.

Solution

Substituting the values for *r*, *n*, *a*, and *b* in the formula for the $(r + 1)$st term yields

$$\binom{10}{4} x^{10-4} y^4 = 210x^6y^4.$$

Introduction to Mathematical Induction

- With mathematical induction we are able to generalize that

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$

- Mathematical induction is a powerful method of proof.
- It is used not only in mathematics, but also in in computer science to prove that programs and basic concepts are correct.

What is the Principle of Mathematical Induction?

PRINCIPLE OF MATHEMATICAL INDUCTION

Let S_n be a statement concerning the positive integer n. Suppose that

1. S_1 is true;
2. for any positive integer k, if S_k is true, then S_{k+1} is also true.

Then, S_n is true for every positive integer n.

Examples of the principle.

- An infinite number of dominoes are lined up.
- An infinite number of rungs on a ladder.

How to Prove by Mathematical Induction

There are two required steps:

PROOF BY MATHEMATICAL INDUCTION

STEP 1: Prove that the statement is true for $n = 1$.
STEP 2: Show that for any positive integer k, if S_k is true, then S_{k+1} is also true.

Let's try to go over these two steps with some examples.

Example

Let S_n represent the statement

$$2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2.$$

Prove that S_n is true for every positive integer.

Solution

Step 1: Show that if the statement S_1 is true, where S_1 is $2^1 = 2^{1+1} - 2$.

since $2 = 4 - 2$, S_1 is a true statement.

Step 2: Show that is S_k is true, then S_{k+1} is also true, where S_k is

and S_{k+1} is $2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$

$$2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2.$$

Example (cont.)

Start with S_k and add 2^{k+1} to each side of the equation. Then, algebraically change the right side to look like the right side of S_{k+1}.

$$\begin{aligned} 2^1 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 2 \\ &= 2^{(k+1)+1} - 2 \end{aligned}$$

The final result is the statement S_{k+1}. Therefore, if S_k is true, then S_{k+1} is also true. The two steps required for a proof by mathematical induction have been completed, so the statement S_n is true for every positive integer n.

Another Example

Prove that if x is a real number between 0 and 1, then for every positive integer n, $0 < x^n < 1$.

Solution

Step1: Here S_1 is the statement if $0 < x < 1$, then $0 < x^1 < 1$, which is true.

Step 2: S_k is the statement if $0 < x < 1$, then $0 < x^k < 1$.

To show that S_k implies that S_{k+1} is true, multiply all three parts of $0 < x^k < 1$ by x to get $x \cdot 0 < x \cdot x^k < x \cdot 1$.

Another Example (cont.)

Simplify to obtain $0 < x^{k+1} < x$.

Since $x < 1$,

$$0 < x^{k+1} < 1,$$

which implies that S_{k+1} is true.

Therefore, if S_k is true, then S_{k+1} is true. Since both steps for a proof by mathematical induction have been completed, the given statement is true for every positive integer n.

Generalized Principle of Mathematical Induction

GENERALIZED PRINCIPLE OF MATHEMATICAL INDUCTION

Let S_n be a statement concerning the positive integer n. Let j be a fixed positive integer. Suppose that

1. S_j is true;
2. for any positive integer k, $k \geq j$, S_k implies S_{k+1}.

Then S_n is true for all positive integers n, where $n \geq j$.

Example: Using the Generalized Principle

Let S_n represent the statement $2^n > 2n + 1$. Show that S_n is true for all values of n such that $n \geq 3$.

Solution

Check that S_n is false for $n = 1$ and $n = 2$.

Step 1: Show that S_n is true for $n = 3$. If $n = 3$, S_3 is

$$2^3 > 2 \cdot 3 + 1, \text{ or}$$

Thus, S_3 is true. $8 > 7.$

One More Example (cont.)

Step 2: Now show that S_k implies S_{k+1}, for $k \geq 3$, where

S_k is $2^k > 2k + 1$ and
S_{k+1} is $2^{k+1} > 2(k + 1) + 1$.

Multiply each side of $2^k > 2k + 1$ by 2, obtaining
$2 \cdot 2^k > 2(2k + 1)$, or
$2^{k+1} > 4k + 2$.

Rewrite $4k + 2$ as $2(k + 1) + 2k$ giving
$2^{k+1} > 2(k + 1) + 2k$.

One More Example (cont.)

Since k is a positive integer greater than 3,
$2k > 1$.

It follows that

$$2^{k+1} > 2(k+1) + 2k > 2(k+1) + 1, \text{ or}$$
$$2^{k+1} > 2(k+1) + 1, \text{ as required.}$$

Thus S_k implies S_{k+1}, and this, together with the fact S_3 is true, shows that S_n is true for every positive integer n greater that or equal to 3.

What have we learned?

We have learned to

1. represent sequences.
2. identify and use arithmetic sequences.
3. identify and use geometric sequences.
4. apply the fundamental counting principle.
5. calculate and apply permutations.
6. calculate and apply combinations.
7. derive the binomial theorem.

What have we learned? (Cont.)

8. use the binomial theorem.
9. apply Pascal's triangle.
10. use mathematical induction to prove statements.
11. apply the generalized principle of mathematical induction.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition