MAC 1140

Module 1
Introduction to Functions and Graphs

Learning Objectives

Upon completing this module, you should be able to

1. recognize common sets of numbers.
2. understand scientific notation and use it in applications.
3. apply problem-solving strategies.
4. analyze one-variable data.
5. find the domain and range of a relation.
6. graph a relation in the xy-plane.
7. calculate the distance between two points.
8. find the midpoint of a line segment.
9. graph equations with a calculator.
10. understand function notation.

Learning Objectives

11. represent a function in five different ways.
12. define a function formally.
13. identify the domain and range of a function.
14. identify functions.
15. identify and use constant and linear functions.
16. interpret slope as a rate of change.
17. identify and use nonlinear functions.
18. recognize linear and nonlinear data.
19. use and interpret average rate of change.
20. calculate the difference quotient.

Introduction to Functions and Graphs

There are four sections in this module:

Let's get started by recognizing some common sets of numbers.

What is the difference between Natural Numbers and Integers?

- Natural Numbers (or counting numbers) are numbers in the set $N = \{1, 2, 3, ...\}$.
- Integers are numbers in the set $I = \{... -3, -2, -1, 0, 1, 2, 3, ...\}$.

What are Rational Numbers?

- Rational Numbers are real numbers which can be expressed as the ratio of two integers *p*/*q* where *q* - 0

Examples: 3 = 3/1 -5 = -10/2 0 = 0/2

0.5 = ½ 0.52 = 52/100 0.333… = 1/3

Note that:
- Every integer is a rational number.
- Rational numbers can be expressed as decimals which either terminate (end) or repeat a sequence of digits.

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What are Irrational Numbers?

- Irrational Numbers are real numbers which are not rational numbers.
- Irrational numbers Cannot be expressed as the ratio of two integers.
- Have a decimal representation which does not terminate and does not repeat a sequence of digits.

Examples:

$$\sqrt{2}, \quad \sqrt[3]{5}, \quad \pi, \quad 0.01001000100001....$$

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Classifying Real Numbers

- Classify each number as one or more of the following: natural number, integer, rational number, irrational number.

$$\sqrt{25}, \quad \sqrt[3]{8}, \quad 3.14, \quad .01010101..., \quad \frac{22}{7}, \quad -\sqrt{11}$$

$\sqrt{25} = 5$ so it is a natural number, integer, rational number

$\sqrt[3]{8} = 2$ so it is a natural number, integer, rational number

3.14, $.01010101....$, and $\frac{22}{7}$ are rational numbers.

$-\sqrt{11}$ is an irrational number.

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Let's Look at Scientific Notation

- A real number *r* is in scientific notation when *r* is written as *c* x 10^n, where and *n* is an integer.

Examples:

- The distance to the sun is 93,000,000 mi.
- In scientific notation this is 9.3 x 10^7 mi.

- The size of a typical virus is .000005cm.
- In scientific notation this is 5 x 10^{-6} cm.

Example

Example 1

Evaluate (5 x 10^6) (3 x 10^{-4}), writing the result in scientific notation and in standard form.

(5 x 10^6) (3 x 10^{-4}) = (5 x 3) x (10^6 x 10^{-4})
= 15 x $10^{6 + (-4)}$
= 15 x 10^2
= 1.5 x 10^3 (scientific notation)
= 1500 (standard form)

Another Example

Example 2

Evaluate $\frac{5\times10^6}{2\times10^{-4}}$, writing the answer in scientific notation and in standard form.

$$\frac{5\times10^6}{2\times10^{-4}} = \frac{5}{2}\times\frac{10^6}{10^{-4}} = 2.5\times10^{6-(-4)} = 2.5\times10^{10}\text{ (scientific notation)}$$

$$= 25{,}000{,}000{,}000\text{ (standard form)}$$

Problem-Solving Strategies

Problem:

A rectangular sheet of aluminum foil is 20 centimeters by 30 centimeters and weighs 4.86 grams. If 1 cubic centimeter of foil weighs 2.7 grams, find the thickness of the foil.

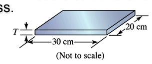
- Possible Solution Strategies
  - Make a sketch.
  - Apply formulas.

Example

Problem:

A rectangular sheet of aluminum foil is 20 centimeters by 30 centimeters and weighs 4.86 grams. If 1 cubic centimeter of aluminum foil weighs 2.70 grams, find the thickness.

Solution:



Start by making a sketch of a rectangular sheet of aluminum, as shown above. Since Volume = Area x Thickness we need to find Volume and Area. Then we will calculate the Thickness by Thickness = Volume/Area

Because the foil weighs 4.86 grams and each 2.70 grams equals 1 cubic centimeter, the volume of the foil is 4.86/2.70 = 1.8 cm^3

The foil is rectangular with an area of 20 centimeters x 30 centimeters = 600 cm^2. The thickness is 1.8 cm^3/600 cm^2 =.003 cm

Analyzing One Variable Data

- Given the numbers -5, 50, 8, 2.5, -7.8, 3.5 find the maximum number, minimum number, range, median, and mean.
  - Arranging the numbers in numerical order yields
    -7.8, -5, 2.5, 3.5, 8, 50.
  - Minimum value is -7.8; maximum value is 50.
  - Range is 50 – (-7.8) = 57.8
  - Median is the middle number. Since there is an even number of numbers, the median is the average of 2.5 and 3.5. The value is 3.
  - Mean is the average of all the six numbers. The value is 8.53.

What is a Relation? What are Domain and Range?

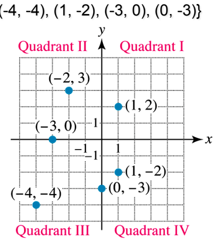
- A relation is a set of ordered pairs.
- If we denote the ordered pairs by (x, y)
  - The set of all x - values is the DOMAIN.
  - The set of all y - values is the RANGE.

Example

- The relation {(1, 2), (-2, 3), (-4, -4), (1, -2), (-3,0), (0, -3)} has domain D = {-4, -3, -2, 0, 1} and range R = {-4, -3, -2, 0, 2, 3}

How to Represent a Relation in a Graph?

The relation {(1, 2), (-2, 3), (-4, -4), (1, -2), (-3, 0), (0, -3)} has the following graph:

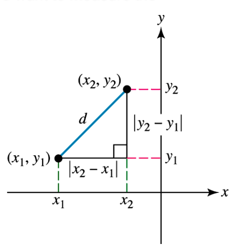


When Do We Use the Distance Formula?

We use the distance formula when we want to measure the distance between two points.

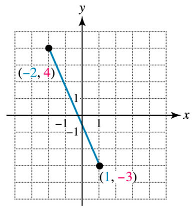
The distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example of Using the Distance Formula

Use the distance formula to find the distance between the two points (-2, 4) and (1, -3).



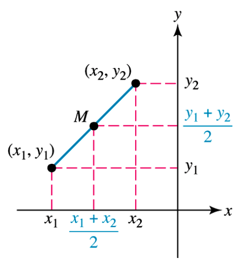
$$d=\sqrt{(1-(-2))^2+(-3-4)^2}=\sqrt{3^2+(-7)^2}=\sqrt{9+49}=\sqrt{58}\approx 7.62$$

Midpoint Formula

The midpoint of the segment with endpoints (x_1, y_1) and (x_2, y_2)

in the *xy*-plane is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

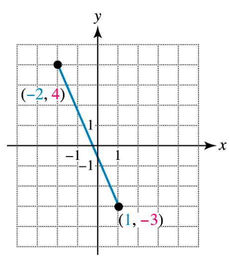


Example of Using the Midpoint Formula

Use the midpoint formula to find the midpoint of the segment with endpoints (-2, 4) and (1, -3).

Midpoint is:

$$\left(\frac{-2+1}{2}, \frac{4+(-3)}{2}\right)=\left(\frac{-1}{2}, \frac{1}{2}\right)$$



Is Function a Relation?

- Recall that a relation is a set of ordered pairs (x,y) .
- If we think of values of *x* as being inputs and values of *y* as being outputs, a function is a relation such that
  - for each input there is exactly one output.

This is symbolized by *output* = *f*(input) or
y = *f*(*x*)

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Function Notation

- *y* = *f*(*x*)
  - Is pronounced "*y* is a function of *x*."
  - Means that given a value of *x* (input), there is exactly one corresponding value of *y* (output).
  - *x* is called the independent variable as it represents inputs, and *y* is called the dependent variable as it represents outputs.
  - Note that: *f*(*x*) is NOT *f* multiplied by *x*. f is NOT a variable, but the name of a function (the name of a relationship between variables).

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What are Domain and Range?

- The set of all meaningful inputs is called the DOMAIN of the function.
- The set of corresponding outputs is called the RANGE of the function.

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What is a Function?

- A **function** is a relation in which each element of the domain corresponds to exactly one element in the range.

Here is an Example

- Suppose a car travels at 70 miles per hour. Let y be the distance the car travels in x hours. Then $y = 70x$.
- Since for each value of x (that is the time in hours the car travels) there is just one corresponding value of y (that is the distance traveled), y is a function of x and we write

  $y = f(x) = 70x$
- Evaluate f(3) and interpret.
  - f(3) = 70(3) = 210. This means that the car travels 210 miles in 3 hours.

Here is Another Example

Given the following data, is y a function of x?

Input x	3	4	8
Output y	6	6	- 5

Note: The data in the table can be written as the set of ordered pairs {(3,6), (4,6), (8, -5)}.

Yes, y is a function of x, because for each value of x, there is just one corresponding value of y. Using function notation we write $f(3) = 6$; $f(4) = 6$; $f(8) = -5$.

One More Example

- Undergraduate Classification at Study-Hard University (SHU) is a function of Hours Earned. We can write this in function notation as $C = f(H)$.
- Why is C a function of H?
  - For each value of H there is exactly one corresponding value of C.
  - In other words, for each input there is exactly one corresponding output.

One More Example (Cont.)

Here is the classification of students at SHU (from catalogue):

No student may be classified as a sophomore until after earning at least 30 semester hours.

No student may be classified as a junior until after earning at least 60 hours.

No student may be classified as a senior until after earning at least 90 hours.

One More Example (Cont.)

Remember **$C = f(H)$**
Evaluate $f(20)$, $f(30)$, $f(0)$, $f(20)$ and $f(61)$:

- $f(20)$ = **Freshman**
- $f(30)$ = **Sophomore**
- $f(0)$ = **Freshman**
- $f(61)$ = **Junior**

- What is the domain of f?
- What is the **range** of f?

One More Example (Cont.)

Domain of f is the set of non-negative integers $\{0,1,2,3,4...\}$

- Alternatively, some individuals say the domain is the set of positive rational numbers, since technically one could earn a fractional number of hours if they transferred in some quarter hours. For example, 4 quarter hours = 2 2/3 semester hours.
- Some might say the domain is the set of non-negative real numbers $[0,\infty)$, but this set includes irrational numbers. It is impossible to earn an irrational number of credit hours. For example, one could not earn $\sqrt{2}$ hours.

Range of *f* is {Fr, Soph, Jr, Sr}

Identifying Functions

- Referring to the previous example concerning SHU, is hours earned a function of classification? That is, is $H = f(C)$? Explain why or why not.
- Is classification a function of years spent at SHU? Why or why not?
- Given $x = y^2$, is y a function of x? Why or why not?
- Given $x = y^2$, is x a function of y? Why or why not?
- Given $y = x^2 + 7$, is y a function of x? Why, why not?

Identifying Functions (Cont.)

- Is hours earned a function of classification? That is, is $H = f(C)$?
- That is, for each value of C is there just one corresponding value of H?
  - No. One example is
    - if C = Freshman, then H could be 3 or 10 (or lots of other values for that matter)

Identifying Functions (Cont.)

- Is classification a function of years spent at SHU? That is, is $C = f(Y)$?
- That is, for each value of Y is there just one corresponding value of C?
  - No. One example is
    - if $Y = 4$, then C could be Sr. or Jr. It could be Jr if a student was a part time student and full loads were not taken.

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Identifying Functions (Cont.)

- Given $x = y^2$, is y a function of x?
- That is, given a value of x, is there just one corresponding value of y?
  - No, if $x = 4$, then $y = 2$ or $y = -2$.

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Identifying Functions (Cont.)

- Given $x = y^2$, is x a function of y?
- That is, given a value of y, is there just one corresponding value of x?
  - Yes, given a value of y, there is just one corresponding value of x, namely y^2.

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Identifying Functions (Cont.)

- Given $y = x^2 + 7$, is y a function of x?
- That is, given a value of x, is there just one corresponding value of y?
  - Yes, given a value of x, there is just one corresponding value of y, namely $x^2 + 7$.

Five Ways to Represent a Function

- Verbally
- Numerically
- Diagrammaticly
- Symbolically
- Graphically

Verbal Representation

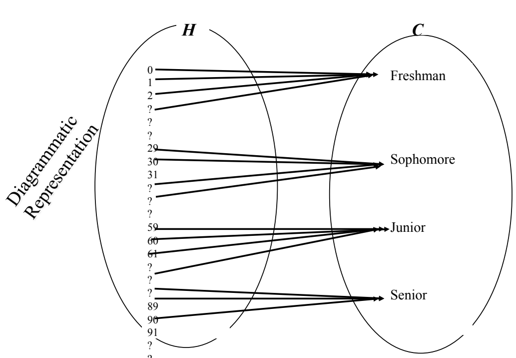
- Referring to the previous example:
  - If you have less than 30 hours, you are a freshman.
  - If you have 30 or more hours, but less than 60 hours, you are a sophomore.
  - If you have 60 or more hours, but less than 90 hours, you are a junior.
  - If you have 90 or more hours, you are a senior.

Numeric Representation

H	*C*
0	Freshman
1	Freshman
? ? ? ?	
29	Freshman
30	Sophomore
31	Sophomore
? ? ?	
59	Sophomore
60	Junior
61	Junior
? ? ?	
89	Junior
90	Senior
91	Senior
? ? ?	

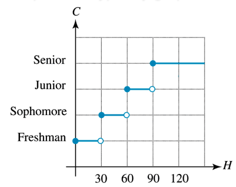
Symbolic Representation

$$C = f(H) = \begin{cases} \text{Freshman} & \text{if } 0 \le H < 30 \\ \text{Sopho} & \text{if } 30 \le H < 60 \\ \text{Junior} & \text{if } 60 \le H < 90 \\ \text{Senior} & \text{if } H \ge 90 \end{cases}$$



Graphical Representation

In this graph the domain is considered to be $[0, \infty)$ instead of {0,1,2,3…}, and note that inputs are typically graphed on the horizontal axis and outputs are typically graphed on the vertical axis.



Vertical Line Test

Another way to determine if a graph represents a function, simply visualize vertical lines in the *xy*-plane. If each vertical line intersects a graph at no more than one point, then it is the graph of a function.

What is a Constant Function?

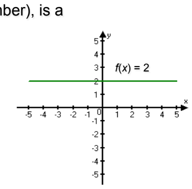
- A function *f* represented by *f*(*x*) = *b*, where *b* is a constant (fixed number), is a constant function.

Examples:

$$f(x) = 2$$

$$f(x) = \frac{-1}{2}$$

$$f(x) = \sqrt{2}$$



Note: Graph of a constant function is a horizontal line.

What is a Linear Function?

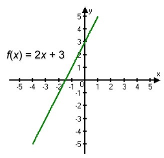
- A function f represented by $f(x) = ax + b$, where a and b are constants, is a linear function.

Examples:

$f(x) = 2x + 3$ (Note: $a = 2$ and $b = 3$)

$f(x) = -5x - \frac{1}{2}$ $\left(\text{Note: } a = -5 \text{ and } b = -\frac{1}{2}\right)$

$f(x) = 2$ (Note: $a = 0$ and $b = 2$)



Note that a $f(x) = 2$ is both a linear function and a constant function.
A constant function is a special case of a linear function.

Rate of Change of a Linear Function

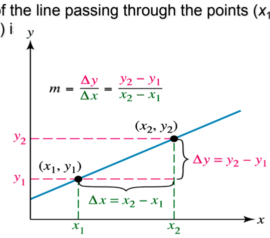
x	y
-2	-1
-1	1
0	3
1	5
2	7
3	9

Table of values for $f(x) = 2x + 3$.

Note throughout the table, as x increases by 1 unit, y increases by 2 units. In other words, the RATE OF CHANGE of y with respect to x is constantly 2 throughout the table. Since the rate of change of y with respect to x is constant, the function is LINEAR. Another name for rate of change of a linear function is SLOPE.

The Slope of a Line

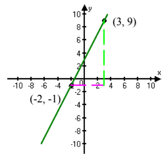
The slope m of the line passing through the points (x_1, y_1) and (x_2, y_2) i



Example of Calculation of Slope

- Find the slope of the line passing through the points (-2, -1) and (3, 9).

$$m = \frac{\Delta y}{\Delta x} = \frac{9-(-1)}{3-(-2)} = \frac{10}{5} = 2$$

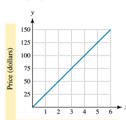


The slope being 2 means that for each unit x increases, the corresponding increase in *y* is 2. The rate of change of *y* with respect to x is 2/1 or 2.

Example of a Linear Function

- The table and corresponding graph show the price *y* of *x* tons of landscape rock.

X (tons)	*y* (price in dollars)
1	25
2	50
3	75
4	100

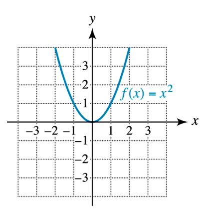


y is a linear function of *x* and the slope is $\frac{\Delta y}{\Delta x} = \frac{50-25}{2-1} = 25$

The rate of change of price *y* with respect to tonage *x* is 25 to 1. This means that for an increase of 1 ton of rock the price increases by $25.

Example of a Nonlinear Function

x	y
0	0
1	1
2	4



- Table of values for $f(x) = x^2$
- Note that as *x* increases from 0 to 1, *y* increases by 1 unit; while as x increases from 1 to 2, *y* increases by 3 units. 1 does not equal 3. This function does NOT have a CONSTANT RATE OF CHANGE of *y* with respect to *x*, so the function is NOT LINEAR. Note that the graph is not a line.

Average Rate of Change

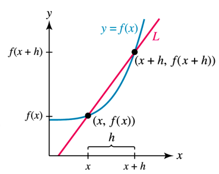
- Let (x_1, y_1) and (x_2, y_2) be distinct points on the graph of a function *f*. The average rate of change of *f* from x_1 to x_2 is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Note that the average rate of change of f from x_1 to x_2 is the slope of the line passing through (x_1, y_1) and (x_2, y_2)

What is the Difference Quotient?

- The difference quotient of a function *f* is an expression of the form $\dfrac{f(x+h) - f(x)}{h}$ where *h* is not 0.



Note that a difference quotient is actually an average rate of change.

What Have We Learned?

We have learned to:

1. recognize common sets of numbers.
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6. graph a relation in the xy-plane.
7. calculate the distance between two points.
8. find the midpoint of a line segment.
9. graph equations with a calculator.
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20. calculate the difference quotient.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 4th Edition
- Rockswold, Gary, Precalculus with Modeling and Visualization, 5th Edition