

MAC 1140
Module 2
Linear Functions and Equations

Learning Objectives

Upon completing this module, you should be able to

1. recognize exact and approximate methods.
2. identify the graph of a linear function.
3. identify a table of values for a linear function.
4. model data with a linear function.
5. use linear regression to model data.
6. write the point-slope and slope-intercept forms for a line.
7. find the intercepts of a line.
8. write equations for horizontal, vertical, parallel, and perpendicular lines.
9. model data with lines and linear functions.

<http://faculty.valenciacollege.edu/ashaw17/> 2

Learning Objectives

10. understand interpolation and extrapolation.
11. use direct variation to solve problems.
12. understand basic terminology related to equations.
13. recognize linear equations symbolically.
14. solve linear equations graphically and numerically.
15. understand the intermediate value property.
16. solve problem involving percentages.
17. apply problem-solving strategies.

<http://faculty.valenciacollege.edu/ashaw17/> 3

Linear Functions and Equations

There are three sections in this module:

- 2.1 Equations of Lines
- 2.2 Linear Equations
- 2.4 More Modeling with Functions

<http://faculty.valenciacollege.edu/ashaw17/>

4

Let's get started by looking at the difference between an exact model and an approximate model.

<http://faculty.valenciacollege.edu/ashaw17/>

5

Example of an Exact Model

- The function $f(x) = 2.1x - 7$ models the data in table exactly.

x	-1	0	1	2
y	-9.1	-7	-4.9	-2.8

Note that:

- $f(-1) = 2.1(-1) - 7 = -9.1$ (Agrees with value in table)
- $f(0) = 2.1(0) - 7 = -7$ (Agrees with value in table)
- $f(1) = 2.1(1) - 7 = -4.9$ (Agrees with value in table)
- $f(2) = 2.1(2) - 7 = -2.8$ (Agrees with value in table)

<http://faculty.valenciacollege.edu/ashaw17/>

6

Example of an Approximate Model

The function $f(x) = 5x + 2.1$ models the data in table **approximately**.

x	1	0	1
y	2.9	2.1	7

Note that:

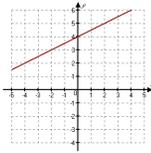
- $f(1) = 5(1) + 2.1 = 2.9$ (Agrees with value in table)
- $f(0) = 5(0) + 2.1 = 2.1$ (Agrees with value in table)
- $f(1) = 5(1) + 2.1 = 7.1$ (Value is **approximately** the value in the table, but not **exactly**.)

<http://faculty.valenciacollege.edu/ashaw17/>

7

What Can You Identify from the Graph of a Linear Function?

• Observe the graph of $f(x) = \frac{1}{2}x + 4$



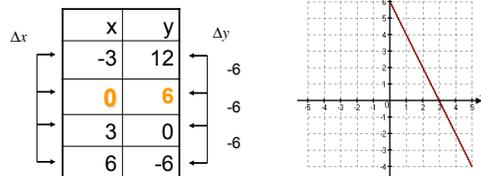
- Note on the graph that as x increases by 2 units, y increases by 1 unit so the slope $\frac{\Delta y}{\Delta x}$ is $\frac{1}{2}$.
- y -intercept (point where graph crosses the y -axis) is $(0, 4)$.
- Equation is $f(x) = \frac{1}{2}x + 4$

<http://faculty.valenciacollege.edu/ashaw17/>

8

Table of Values for Linear Functions

- $f(x) = -2x + 6$
- Note that as x increases by 3 units, y decreases by 6 units so the slope $\frac{\Delta y}{\Delta x}$ is $-6 / 3 = -2$

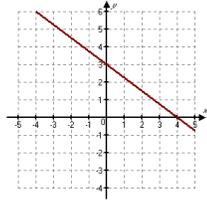


<http://faculty.valenciacollege.edu/ashaw17/>

9

Let's Practice Writing an Equation of a Linear Function

- What is the slope?
 - As x increases by 4 units, y decreases by 3 units so the slope is $-3/4$
- What is the y -intercept?
 - The graph crosses the y axis at $(0,3)$, so the y -intercept is 3 .
- What is the equation?
 - Equation is $f(x) = (-3/4)x + 3$

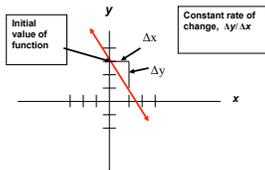


<http://faculty.valenciacollege.edu/ashaw17/>

10

Modeling with Linear Functions

- A linear function, $f(x) = mx + b$, has a constant rate of change, that is a constant slope.
- $f(0) = m(0) + b = b$. When the input of the function is 0, the output is b . So the y -intercept b is sometimes called the initial value of the function.



Note that a linear function is always a straight line.

<http://faculty.valenciacollege.edu/ashaw17/>

11

The Linear Function Model

- To model a quantity that is changing at a constant rate, the following may be used.

$$f(x) = (\text{constant rate of change})x + \text{initial amount}$$

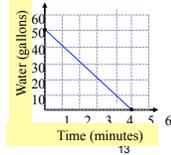
- Because
 - constant rate of change corresponds to the slope
 - initial amount corresponds to the y -intercept
 this is simply $f(x) = mx + b$

<http://faculty.valenciacollege.edu/ashaw17/>

12

Example of Modeling with a Linear Function

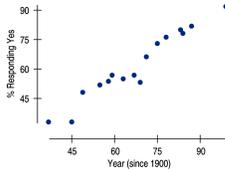
- A 50-gallon tank is initially full of water and being drained at a constant rate of 10 gallons per minute. Write a formula that models the number of gallons of water in the tank after x minutes.
- The water in the tank is changing at a constant rate, so the linear function model $f(x) = (\text{constant rate of change})x + \text{initial amount}$ applies.
- So $f(x) = (-10 \text{ gal/min})(x \text{ min}) + 50 \text{ gal}$. Without specifically writing the units, this is $f(x) = -10x + 50$



<http://faculty.valenciacollege.edu/ashaw17/>

What is a Scatterplot?

- Scatterplot may be the most common and most effective display for data.
- In a scatterplot, you can see patterns, trends, relationships, and even the occasional extraordinary value sitting apart from the others.



• This figure shows a **positive association** between the year since 1900 and the % of people who say they would vote for a woman president.

• As the years have passed, the percentage who would vote for a woman has increased.

<http://faculty.valenciacollege.edu/ashaw17/>

The Linear Model

- The **linear model** is just an equation of a straight line through the data.
- The points in the scatterplot don't all line up, but a straight line can summarize the general pattern.
- The linear model can help us understand how the values are **associated**.

<http://faculty.valenciacollege.edu/ashaw17/>

The Linear Model and the Predicted Value

- The model won't be perfect, regardless of the line we draw.
- Some points will be above the line and some will be below.
- The estimate made from a model is the **predicted value**.

<http://faculty.valenciacollege.edu/ashaw17/>

16

The Least Square Regression Line

- We approximate our **linear model** with $f(x) = ax + b$
- This model says that our *predictions* from our model follow a **straight line**.
- If the model is a good one, the data values will scatter closely around it.

<http://faculty.valenciacollege.edu/ashaw17/>

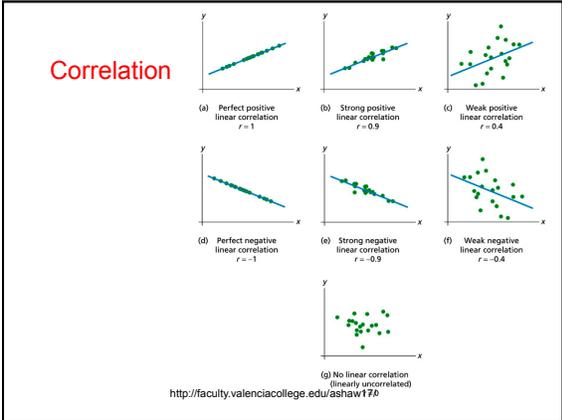
17

What is Correlation?

- **Regression** and **correlation** are closely related. Correlation measures the strength of the **linear association** between two variables: x and y .
- **Correlation** treats x and y symmetrically:
 - The correlation of x with y is the same as the correlation of y with x . Correlation has no units.
- **Correlation** is always between -1 and $+1$.
 - Correlation can be exactly equal to -1 or $+1$, but these values are unusual in real data because they mean that all the data points fall *exactly* on a single straight line.
 - A **correlation near zero** corresponds to a **weak linear association**.

<http://faculty.valenciacollege.edu/ashaw17/>

18



How to Use Calculator to Find the Correlation Coefficient, r ?

Here are the steps:

- Under **STAT EDIT** choose "1: Edit," enter the values for both variables: x and y , under column title L1 and L2.
- Now, hit **2nd CATALOG** (on the zero key) to turn on the correlation coefficient's calculation feature of your TI-83/84+ calculator. Scroll down until you find **Dia9nosticOn**. Hit **ENTER** twice. It should say Done. (From now on, your calculator will be able to find correlation, unless the battery is dead.)
- Under **STAT CALC** choose "4: LinReg(ax+b)," enter "L1,L2" and hit **ENTER**. (You now see not only " r ," but also the linear regression line $y=ax+b$.)

Please go through the Calculator Tutorial on this topic.

<http://faculty.valenciacollege.edu/ashaw17/> 20

What is the Point-Slope Form of the Equation of a Line?

The line with slope m passing through the point (x_1, y_1) has equation

$$y = m(x - x_1) + y_1$$

or

$$y - y_1 = m(x - x_1)$$

<http://faculty.valenciacollege.edu/ashaw17/> 21

How to Write the Equation of the Line Passing Through the Points (-4, 2) and (3, -5)

- To write the equation of the line using **point-slope form**
 $y = m(x - x_1) + y_1$
the **slope** m and a **point** (x_1, y_1) are needed.
 - Let $(x_1, y_1) = (3, -5)$.
 - Calculate m using the two given points.
$$m = \frac{\Delta y}{\Delta x} = \frac{-5 - 2}{3 - (-4)} = \frac{-7}{7} = -1$$
 - Equation is $y = -1(x - 3) + (-5)$
 - This simplifies to $y = -x + 3 + (-5)$
 $y = -x - 2$

<http://faculty.valenciacollege.edu/ashaw17/> 22

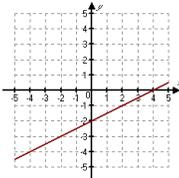
Slope-Intercept Form

- The line with **slope** m and **y-intercept** b is given by
$$y = m x + b$$

<http://faculty.valenciacollege.edu/ashaw17/> 23

How to Write the Equation of a line passing through the point (0,-2) with slope $\frac{1}{2}$?

- Since the point (0, -2) has an x-coordinate of 0, the point is a y-intercept. Thus $b = -2$
- Using **slope-intercept form**
 $y = m x + b$
the equation is
 $y = (\frac{1}{2}) x - 2$



<http://faculty.valenciacollege.edu/ashaw17/> 24

What is the Standard Form for the Equation of a Line?

$$ax + by = c$$

is standard form for the equation of a line.

<http://faculty.valenciacollege.edu/ashaw17/>

25

How to Find x-Intercept and y-intercept of $2x-3y=6$?

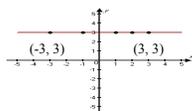
- To find the x-intercept, let $y = 0$ and solve for x .
 - $2x - 3(0) = 6$
 - $2x = 6$
 - $x = 3$
- To find the y-intercept, let $x = 0$ and solve for y .
 - $2(0) - 3y = 6$
 - $-3y = 6$
 - $y = -2$

<http://faculty.valenciacollege.edu/ashaw17/>

26

What are the Characteristics of Horizontal Lines?

- Slope is 0, since $\Delta y = 0$ and $m = \Delta y / \Delta x$
- Equation is: $y = mx + b$
 - $y = (0)x + b$
 - $y = b$ where b is the y-intercept
- Example: $y = 3$ (or $0x + y = 3$)



Note that regardless of the value of x , the value of y is always 3.

<http://faculty.valenciacollege.edu/ashaw17/>

27

What are Interpolation and Extrapolation?

- The U.S. sales of Toyota vehicles in millions is listed below.

Year	1998	2000	2002
Vehicles	1.4	1.6	1.8

- Writing the equation of the line passing through these three points yields the following equation which models the data exactly.

$$y = .1x - 198.4$$
- Example of Interpolation:** Using the model to predict the sales in the year 1999 we have $y = .1(1999) - 198.4 = 1.5$. This is an example of interpolation because 1999 lies between 1998 and 2002.
- Example of Extrapolation:** Using the model to predict the sales in the year 2004 we have $y = .1(2004) - 198.4 = 2$. This is an example of extrapolation because 2004 does not lie between 1998 and 2002.

<http://faculty.valenciacollege.edu/ashaw17/>

31

Direct Variation and Constant of Proportionality

- Let x and y denote two quantities. Then y is **directly proportional** to x , or y **varies directly** with x , if there exists a **nonzero number k** such that

$$y = kx$$
 k is called the **constant of proportionality** or the constant of variation.
- Example:** Suppose the sales tax is 6%. Let x represent the amount of a purchase and let y represent the sales tax on the purchase. Then y is **directly proportional to x** with **constant of proportionality .06**.

$$y = .06x$$

<http://faculty.valenciacollege.edu/ashaw17/>

32

Here is an Example of Solving a Problem Involving Direct Variation

- Sales tax y on a purchase is **directly proportional** to the amount of the purchase x with the constant of proportionality being the % of sales tax charged. If the sales tax on a \$110 purchase was \$8, **what is the sales tax rate** (constant of proportionality)? Find the tax on a \$90 purchase.
 - Since y is **directly proportional to x** , $y = kx$. To **find k** , we are given that when $x = \$110$, $y = \$8$, so
 - $8 = 110k$ and $k = 8/110 = .0727$. Thus the sales tax rate is 7.27%.
 - To find the tax on a \$90 purchase,
 - $y = .0727x$ so when $x = \$90$, $y = .0727(\$90) = \6.54

<http://faculty.valenciacollege.edu/ashaw17/>

33

Here is Another Example

- Suppose that a car travels at 70 miles per hour for x hours. Then the distance y that the car travels is directly proportional to x . Find the constant of proportionality. How far does the car travel in two hours?
 - Since y is directly proportional to x , $y = kx$.
 - Specifically y mi = (70 mi/hr) (x hr)
 - Without writing units, $y = 70x$ and the constant of proportionality is 70.
 - To predict how far the car travels in two hours,
 - $y = 70x$
 - When $x = 2$, $y = 140$, so the car travels 140 miles in two hours.

<http://faculty.valenciacollege.edu/ashaw17/>

34

Contradiction and Identity

- Contradiction – An equation for which there is no solution.
 - Example: $2x + 3 = 5 + 4x - 2x$
 - Simplifies to $2x + 3 = 2x + 5$
 - Simplifies to $3 = 5$
 - FALSE statement – there are no values of x for which $3 = 5$. The equation has NO SOLUTION.
- Identity – An equation for which every meaningful value of the variable is a solution.
 - Example: $2x + 3 = 3 + 4x - 2x$
 - Simplifies to $2x + 3 = 2x + 3$
 - Simplifies to $3 = 3$
 - TRUE statement – no matter the value of x , the statement $3 = 3$ is true. The solution is ALL REAL NUMBERS.

<http://faculty.valenciacollege.edu/ashaw17/>

35

What is a Conditional Equation?

- Conditional Equation – An equation that is satisfied by some, but not all, values of the variable.
 - Example: $x^2 = 1$
 - Solutions of the equation are:
 - » $x = -1, x = 1$

<http://faculty.valenciacollege.edu/ashaw17/>

36

Linear Equations in One Variable

- A linear equation in one variable is an equation that can be written in the form $ax + b = 0$ where a and b are real numbers with $a \neq 0$. (Note the power of x is always 1.)
- Examples of linear equations in one variable:
 - $5x + 4 = 2 + 3x$ simplifies to $2x + 2 = 0$
Note the power of x is always 1.
 - $-1(x - 3) + 4(2x + 1) = 5$ simplifies to $7x + 2 = 0$
Note the power of x is always 1.
- Examples of equations in one variable which are not linear:
 - $x^2 = 1$ (Note the power of x is NOT 1.)
 - $\frac{1}{x-1} + x = 0$ (Note the power of x is NOT always 1.)

<http://faculty.valenciacollege.edu/ashaw17/>

37

How to Solve a Linear Equations Symbolically?

- Solve $-1(x - 3) + 4(2x + 1) = 5$ for x
 - $-1x + 3 + 8x + 4 = 5$
 - $7x + 7 = 5$
 - $7x = 5 - 7$
 - $7x = -2$
 - $x = -2/7$ "Exact Solution"
- Linear Equations can always be solved symbolically and will produce an EXACT SOLUTION. The solution procedure is to isolate the variable on the left in a series of steps in which the same quantity is added to or subtracted from each side and/or each side is multiplied or divided by the same non-zero quantity. This is true because of the addition and multiplication properties of equality.

<http://faculty.valenciacollege.edu/ashaw17/>

38

How to Solve a Linear Equation Involving Fractions Symbolically?

- Solve $\frac{x-1}{3} + 5 = \frac{1}{4}$
- Solution Process:
 - $12\left(\frac{x-1}{3} + 5\right) = 12\left(\frac{1}{4}\right)$
 - $4(x-1) + 60 = 3$
 - $4x - 4 + 60 = 3$
 - $4x + 56 = 3$
 - $4x = 3 - 56$
 - $4x = -53$
 - $x = \frac{-53}{4} = -13.25$
- When solving a linear equation involving fractions, it is often helpful to multiply both sides by the least common denominator of all of the denominators in the equation.
- The least common denominator of 3 and 4 is 12.
- Note that this is another "Exact Solution."

<http://faculty.valenciacollege.edu/ashaw17/>

39

How to Solve a Linear Equation Graphically?

- Solve $\frac{x-1}{3} + 5 = \frac{1}{4}$
- Solution Process:
 - Graph $y_1 = \frac{x-1}{3} + 5$
 - $y_2 = \frac{1}{4}$
 - in a window in which the graphs intersect.

[-20, 5, 1] by [-2, 2, 1]

<http://faculty.valenciacollege.edu/ashaw17/> 40

How to Solve a Linear Equation Graphically?(Cont.)

- Locate **points of intersection**. x -coordinates of points of intersection are **solutions** to the equation.

[-20, 5 1] by [-2, 2, 1]

The solution to the equation is -13.25.

This agrees exactly with the solution produced from the symbolic method. Sometimes a graphical method will produce only an **approximate solution**.

<http://faculty.valenciacollege.edu/ashaw17/> 41

Another Example

- Solve $3x - 17 = \sqrt{2}x - 4$

Plot1 Plot2 Plot3

$\sqrt{1} \text{ } 3x - 17$

$\sqrt{2} \text{ } \sqrt{2}x - 4$

$\sqrt{3} =$

$\sqrt{4} =$

$\sqrt{5} =$

$\sqrt{6} =$

$\sqrt{7} =$

[-2, 15, 1] by [-2, 15, 1]

Approximate solution (to the nearest hundredth) is 8.20. The exact solution of $\frac{13}{3-\sqrt{2}}$ can be found by solving symbolically.

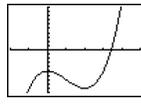
<http://faculty.valenciacollege.edu/ashaw17/> 42

Intermediate Value Property

Let (x_1, y_1) and (x_2, y_2) with $y_1 \neq y_2$ and $x_1 < x_2$ be two points on the graph of a continuous function f . Then, on the interval $x_1 \leq x \leq x_2$, f assumes every value between y_1 and y_2 at least once.

- Below is a table of values for $f(x) = x^3 - 3x^2 - 5$

X	Y
-5	315
-4	191
-3	103
-2	45
-1	11
0	-5
1	-8
2	-5
3	11
4	191



Since f is continuous, and the points $(3, -5)$ and $(4, 11)$ satisfy f , we know that x assumes every value between -5 and 11 at least once. Thus we know that the graph has an x -intercept between $x = 3$ and $x = 4$.

<http://faculty.valenciacollege.edu/ashaw17/>

43

What Are the Four Steps in Modeling with Linear Equations?

- STEP 1:** Read the problem and make sure you understand it. Assign a variable to what you are being asked. If necessary, write other quantities in terms of the variable.
- STEP 2:** Write an equation that relates the quantities described in the problem. You may need to sketch a diagram and refer to known formulas.
- STEP 3:** Solve the equation and determine the solution.
- STEP 4:** Look back and check your solution. Does it seem reasonable?

<http://faculty.valenciacollege.edu/ashaw17/>

44

Example of Modeling with Linear Equations

- In 2 hours an athlete travels 18.5 miles by running at 11 miles per hour and then by running at 9 miles per hour. How long did the athlete run at each speed?
- STEP 1:** We are asked to find the time spent running at each speed. If we let x represent the time in hours running at 11 miles per hour, then $2 - x$ represents the time spent running at 9 miles per hour.
 - x : Time spent running at 11 miles per hour
 - $2 - x$: Time spent running at 9 miles per hour
- STEP 2:** Distance d equals rate r times time t ; that is, $d = rt$. In this example we have two rates (speeds) and two times. The total distance must sum to 18.5 miles.

$$d = r_1t_1 + r_2t_2$$

$$18.5 = 11x + 9(2 - x)$$

<http://faculty.valenciacollege.edu/ashaw17/>

45

Example of Modeling with Linear Equations (Cont.)

- STEP 3: Solving $18.5 = 11x + 9(2 - x)$ symbolically
 $18.5 = 11x + 18 - 9x$
 $18.5 - 18 = 2x$
 $.5 = 2x$
 $x = .5/2$
 $x = .25$

The athlete runs .25 hours (15 minutes) at 11 miles per hour and 1.75 hours (1 hours and 45 minutes) at 9 miles per hour.

- STEP 4: We can check the solution as follows.
 $11(.25) + 9(1.75) = 18.5$ (It checks.)
This sounds reasonable. The average speed was 9.25 mi/hr, that is 18.5 miles/2 hours. Thus the runner would have to run longer at 9 miles per hour than at 11 miles per hour, since 9.25 is closer to 9 than 11.

<http://faculty.valenciacollege.edu/ashaw17/>

46

Let's Try to Solve One More Linear Equations Problem

- Pure water is being added to a 25% solution of 120 milliliters of hydrochloric acid. How much water should be added to reduce it to a 15% mixture?
- STEP 1: We need the amount of water to be added to 120 milliliters of 25% acid to make a 15% solution. Let this amount of water be equal to x .
 x : Amount of pure water to be added
 $x + 120$: Final volume of 15% solution
- STEP 2: The total amount of acid in the solution after adding the water must equal the amount of acid before the water is added. The volume of pure acid after the water is added equals 15% of $x + 120$ milliliters, and the volume of pure acid before the water is added equals 25% of 120 milliliters. So we must solve the equation

$$.15(x + 120) = .25(120)$$

<http://faculty.valenciacollege.edu/ashaw17/>

47

Let's Try to Solve One More Linear Equations Problem (Cont.)

- STEP 3: Solving $.15(x + 120) = .25(120)$ symbolically
 $.15x + 18 = 30$
 $.15x = 12$
 $x = 12/.15$
 $x = 80$ milliliters
- STEP 4: This sounds reasonable. If we added 120 milliliters of water, we would have diluted the acid to half its concentration, which would be 12.5%. It follows that we should not add much as 120 milliliters since we want a 15% solution.

<http://faculty.valenciacollege.edu/ashaw17/>

48

What Have We learned?

We have learned to

1. recognize exact and approximate methods.
2. identify the graph of a linear function.
3. identify a table of values for a linear function.
4. model data with a linear function.
5. use linear regression to model data.
6. write the point-slope and slope-intercept forms for a line.
7. find the intercepts of a line.
8. write equations for horizontal, vertical, parallel, and perpendicular lines.
9. model data with lines and linear functions.

<http://faculty.valenciacollege.edu/ashaw17/>

49

What Have We Learned? (Cont.)

10. understand interpolation and extrapolation.
11. use direct variation to solve problems.
12. understand basic terminology related to equations.
13. recognize linear equations symbolically.
14. solve linear equations graphically and numerically.
15. understand the intermediate value property.
16. solve problem involving percentages.
17. apply problem-solving strategies.

<http://faculty.valenciacollege.edu/ashaw17/>

50

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Rockswold, Gary, Precalculus with Modeling and Visualization, 4th Edition
- Rockswold, Gary, Precalculus with Modeling and Visualization, 5th Edition
- Weiss, Neil A., Introductory Statistics, 8th Edition

<http://faculty.valenciacollege.edu/ashaw17/>

51
