MAC 1140

Module 8
Logarithmic Functions

Learning Objectives

- Upon completing this module, you should be able to

1. evaluate the common logarithmic function.
2. solve basic exponential and logarithmic equations.
3. evaluate logarithms with other bases.
4. solve general exponential and logarithmic equations.
5. apply basic properties of logarithms.
6. use the change of base formula.
7. solve exponential equations.
8. solve logarithmic equations.

Logarithmic Functions

There are three sections in this module:

5.4 Logarithmic Functions and Models
5.5 Properties of Logarithms
5.6 Exponential and Logarithmic Equations

What is the Definition of a Common Logarithmic Function?

- The common logarithm of a positive number x, denoted $\log(x)$, is defined by

  $\log(x) = k$ if and only if $x = 10^k$

  where k is a real number.

- The function given by $f(x) = \log(x)$ is called the common logarithmic function.
- Note that the input x must be positive.

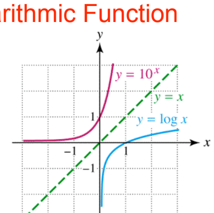
Let's Evaluate Some Common Logarithms

- $\log(10)$
- $\log(100)$
- $\log(1000)$
- $\log(10000)$
- $\log(1/10)$
- $\log(1/100)$
- $\log(1/1000)$
- $\log(1)$

- **1** because $10^1 = 10$
- **2** because $10^2 = 100$
- **3** because $10^3 = 1000$
- **4** because $10^4 = 10000$
- **–1** because $10^{-1} = 1/10$
- **–2** because $10^{-2} = 1/100$
- **–3** because $10^{-3} = 1/1000$
- **0** because $10^0 = 1$

Let's Take a Look at the Graph of a Logarithmic Function

x	$f(x)$
0.01	-2
0.1	-1
1	0
10	1
100	2



Note that the graph of $y = \log(x)$ is the graph of $y = 10^x$ reflected through the line $y = x$. This suggests that these are inverse functions.

What is the Inverse Function of a Common Logarithmic Function?

- Note that the graph of $f(x) = \log(x)$ passes the horizontal line test so it is a one-to-one function and has an inverse function.
- Find the inverse of $y = \log(x)$
- Using the definition of common logarithm to solve for x gives $x = 10^y$
- Interchanging x and y gives
- $y = 10^x$
- Thus, the inverse of $y = \log(x)$ is $y = 10^x$

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What is the Inverse Properties of the Common Logarithmic Function?

- Recall that $f^{-1}(x) = 10^x$ given $f(x) = \log(x)$
- Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}
  - **$\log(10^x) = x$ for all real numbers x.**
- Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f
  - **$10^{\log x} = x$ for any positive number x**

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What is the Definition of a Logarithmic Function with base a?

- The logarithm with base a of a positive number x, denoted by $\log_a(x)$ is defined by
  $\log_a(x) = k$ if and only if $x = a^k$
  where $a > 0$, $a \neq 1$, and k is a real number.
- The function given by $f(x) = \log_a(x)$ is called the logarithmic function with base a.

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What is the Natural Logarithmic Function?

- Logarithmic Functions with Base 10 are called "common logs."
  - log (x) means $\log_{10}(x)$ - *The Common Logarithmic Function*
- Logarithmic Functions with Base e are called "natural logs."
  - ln (x) means $\log_e(x)$ - *The Natural Logarithmic Function*

Let's Evaluate Some Natural Logarithms

- ln (e)
- ln (e^2)
- ln (1)
- $\ln \sqrt{e}$

- ln $(e) = \log_e(e) = 1$ since $e^1 = e$
- $\ln(e^2) = \log_e(e^2) = 2$ since 2 is the exponent that goes on e to produce e^2.
- ln $(1) = \log_e 1 = 0$ since $e^0 = 1$
- 1/2 since 1/2 is the exponent that goes on e to produce $e^{1/2}$

What is the Inverse of a Logarithmic Function with base a?

- Note that the graph of $f(x) = \log_a(x)$ passes the horizontal line test so it is a one-to-one function and has an inverse function.
- Find the inverse of $y = \log_a(x)$
- Using the definition of common logarithm to solve for x gives
- $x = a^y$
- Interchanging x and y gives
- $y = a^x$
- Thus, the inverse of $y = \log_a(x)$ is $y = a^x$

What is the Inverse Properties of a Logarithmic Function with base a?

- Recall that $f^{-1}(x) = a^x$ given $f(x) = \log_a(x)$
- Since $(f \circ f^{-1})(x) = x$ for every x in the domain of f^{-1}
  - **$\log_a(a^x) = x$ for all real numbers x.**
- Since $(f^{-1} \circ f)(x) = x$ for every x in the domain of f
  - **$a^{\log_a x} = x$ for any positive number x**

Let's Try to Solve Some Exponential Equations

- Solve the equation $4^x = 1/64$
- Take the log of both sides to the base 4
  - $\log_4(4^x) = \log_4(1/64)$
- Using the inverse property $\log_a(a^x) = x$, this simplifies to
  - $x = \log_4(1/64)$
- Since 1/64 can be rewritten as 4^{-3}
  - $x = \log_4(4^{-3})$
- Using the inverse property $\log_a(a^x) = x$, this simplifies to
  - $x = -3$

Let's Try to Solve Some Exponential Equations (Cont.)

- Solve the equation $e^x = 15$
- Take the log of both sides to the base e
- $\ln(e^x) = \ln(15)$
- Using the inverse property $\log_a(a^x) = x$ this simplifies to
  - $x = \ln(15)$
- Using the calculator to estimate ln (15)
  - $x \approx 2.71$

Let's Try to Solve Some Logarithmic Equations (Cont.)

- Solve the equation ln (x) = 1.5
- Exponentiate both sides using base e
  - $e^{\ln x} = e^{1.5}$
- Using the inverse property $a^{\log_a x} = x$ this simplifies to
  - $x = e^{1.5}$
- Using the calculator to estimate $e^{1.5}$
  - $x \approx 4.48$

What are the Basic Properties of Logarithms?

Property 1

- $\log_a(1) = 0$ and $\log_a(a) = 1$
  - $a^0 = 1$ and $a^1 = a$
- Note that this property is a direct result of the inverse property $\log_a(a^x) = x$
- Example: log (1) =0 and ln (e) =1

Property 2

- $\log_a(m) + \log_a(n) = \log_a(mn)$
- The sum of logs is the log of the product.
- Example: Let $a = 2$, $m = 4$ and $n = 8$
- $\log_a(m) + \log_a(n) = \log_2(4) + \log_2(8) = 2 + 3$
- $\log_a(mn) = \log_2(4 \cdot 8) = \log_2(32) = 5$

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Property 3

- $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$
- The difference of logs is the log of the quotient.
- Example: Let $a = 2$, $m = 4$ and $n = 8$

$$\log_a m - \log_a n = \log_2 4 - \log_2 8 = 2 - 3 = -1$$

$$\log_a\left(\frac{m}{n}\right) = \log_2\left(\frac{4}{8}\right) = \log_2\left(\frac{1}{2}\right) = -1$$

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Property 4

- $\log_a(m^r) = r\log_a m$
- Example: Let $a = 2$, $m = 4$ and $r = 3$

$$\log_a(m^r) = \log_2(4^3) = \log_2(64) = 6$$

$$r\log_a m = 3\log_2 4 = 3(2) = 6$$

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Example

- Expand the expression. Write without exponents.

$$\log\left(\frac{3x^6}{2y^7}\right)$$

$$\log\left(\frac{3x^6}{2y^7}\right) = \log(3x^6) - \log(2y^7)$$

$$\log 3 + \log(x^6) - (\log 2 + \log(y^7))$$

$$\log 3 + 6\log x - \log 2 - 7\log y$$

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One More Example

- Write as the logarithm of a single expression

$$2\ln x - 4\ln y + \frac{1}{2}\ln z$$

$$2\ln x - 4\ln y + \frac{1}{2}\ln z = \ln\left(x^2\right) - \ln\left(y^4\right) + \ln\left(z^{\frac{1}{2}}\right)$$

$$= \ln\left(\frac{x^2}{y^4}\right) + \ln\left(z^{\frac{1}{2}}\right) = \ln\left(\frac{x^2}{y^4} \times z^{\frac{1}{2}}\right) = \ln\left(\frac{x^2 \times z^{\frac{1}{2}}}{y^4}\right)$$

$$= \ln\left(\frac{x^2 \times \sqrt{z}}{y^4}\right)$$

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What is the Change of Base Formula?

Let x, $a \neq 1$, and $b \neq 1$ be positive real numbers. Then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

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Example of Using the Change of Base Formula?

- Use the change of base formula to evaluate $\log_3 8$

$$\log_3 8 = \frac{\log_{10} 8}{\log_{10} 3} = \frac{\log 8}{\log 3} \approx 1.893$$

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Modeling Compound Interest

- How long does it take money to grow from \$100 to \$200 if invested into an account which compounds quarterly at an annual rate of 5%?
- Must solve for t in the following equation

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

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Modeling Compound Interest (Cont.)

$$200 = 100\left(1 + \frac{.05}{4}\right)^{4t}$$

$2 = (1.0125)^{4t}$ Divide each side by 100

$\log 2 = \log(1.0125)^{4t}$ Take common logarithm of each side

$\log 2 = 4t \log 1.0125$ Property 4: $\log(m^r) = r \log(m)$

$4t \log 1.0125 = \log 2$

$t = \frac{\log 2}{4 \log 1.0125}$ Divide each side by 4log1.0125

$t \approx 13.95$ years Approximate using calculator

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Modeling Compound Interest (Cont.)

Alternatively, we can take natural logarithm of each side instead of taking the common logarithm of each side.

$$200 = 100\left(1+\frac{.05}{4}\right)^{4t}$$

Divide each side by 100

$$2 = (1.0125)^{4t}$$

Take natural logarithm of each side

$$\ln 2 = \ln(1.0125)^{4t}$$

Property 4: $\ln(m^r) = r\ln(m)$

$$\ln 2 = 4t\ln 1.0125$$

Divide each side by 4 ln (1.0125)

$$4t\ln 1.0125 = \ln 2$$

Approximate using calculator

$$t = \frac{\ln 2}{4\ln 1.0125}$$

$$t \approx 13.95 \text{ years}$$

Solve $3(1.2)^x + 2 = 15$ for x symbolically

$$3(1.2)^x = 13$$

Divide each side by 3

$$1.2^x = \frac{13}{3}$$

Take common logarithm of each side

(Could use natural logarithm)

$$\log 1.2^x = \log\left(\frac{13}{3}\right)$$

Property 4: $\log(m^r) = r\log(m)$

$$x\log 1.2 = \log\left(\frac{13}{3}\right)$$

Divide each side by log (1.2)

$$x = \frac{\log\left(\frac{13}{3}\right)}{\log 1.2}$$

Approximate using calculator

$$x \approx 8.04$$

Solve $e^{x+2} = 5^{2x}$ for x symbolically

$$e^{x+2} = 5^{2x}$$

Take natural logarithm of each side

$$\ln\left(e^{x+2}\right) = \ln\left(5^{2x}\right)$$

Property 4: $\ln(m^r) = r\ln(m)$

$$(x+2)\ln e = 2x\ln 5$$

$\ln(e) = 1$

$$x+2 = 2x\ln 5$$

Subtract $2x\ln(5)$ and 2 from each side

$$x - 2x\ln 5 = -2$$

Factor x from left-hand side

$$x(1-2\ln 5) = -2$$

Divide each side by $1 - 2\ln(5)$

$$x = \frac{-2}{1-2\ln 5}$$

Approximate using calculator

$$x \approx .901$$

Solving a Logarithmic Equation Symbolically

- In developing countries there is a relationship between the amount of land a person owns and the average daily calories consumed. This relationship is modeled by the formula $C(x) = 280\ln(x+1) + 1925$ where x is the amount of land owned in acres and

  Source: D. Gregg: *The World Food Problem*
- Determine the number of acres owned by someone whose average intake is 2400 calories per day.
- Must solve for x in the equation

$$280\ln(x+1) + 1925 = 2400$$

Solving a Logarithmic Equation Symbolically (Cont.)

$280\ln(x+1)+1925 = 2400$	
$280\ln(x+1) = 2400-1925$	Subtract 1925 from each side
$280\ln(x+1) = 475$	Divide each side by 280
$\ln(x+1) = \frac{475}{280}$	Exponentiate each side base e
$e^{\ln(x+1)} = e^{\frac{475}{280}}$	Inverse property $e^{\ln k} = k$
$x+1 = e^{\frac{475}{280}}$	Subtract 1 from each side
$x = e^{\frac{475}{280}} - 1$	Approximate using calculator
$x \approx 4.45$	

One More Example

Solve $\log_2(1-x) = 1$ for x

$2^1 = 1-x$	Definition of logarithm $\log_a x = k$ if and only if $x = a^k$
$x+2 = 1$	Add x to both sides of equation
$x = 1-2$	Subtract 2 from both sides of the equation
$x = -1$	

What have we learned?

- We have learned to

1. evaluate the common logarithmic function.
2. solve basic exponential and logarithmic equations.
3. evaluate logarithms with other bases.
4. solve general exponential and logarithmic equations.
5. apply basic properties of logarithms.
6. use the change of base formula.
7. solve exponential equations.
8. solve logarithmic equations.

Credit

- Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:
- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition