MAC 1140

Module 9
System of Equations and Inequalities I

Learning Objectives

Upon completing this module, you should be able to

1. evaluate functions of two variables.
2. apply the method of substitution.
3. apply graphical and numerical methods to system of equations.
4. solve problems involving joint variation.
5. recognize different types of linear systems.
6. apply the elimination method.
7. solve systems of linear and nonlinear inequalities.

Learning Objectives (Cont.)

8. apply the method of substitution.
9. apply graphical and numerical methods to system of equations.
10. solve problems involving joint variation.
11. recognize different types of linear systems.
12. apply the elimination method.
13. solve systems of linear and nonlinear inequalities.

System of Equations and Inequalities I

There are three sections in this module:

9.1 Functions and Equations in Two Variables
9.2 System of Equations and Inequalities in Two Variables
9.3 Systems of Linear Equations in Three Variables

Do We Really Use Functions of Two Variables?

- The answer is YES.
- Many quantities in everyday life depend on more than one variable.

Examples

- Area of a rectangle requires both width and length.
- Heat index is the function of temperature and humidity.
- Wind chill is determined by calculating the temperature and wind speed.
- Grade point average is computed using grades and credit hours.

Let's Take a Look at the Arithmetic Operations

- The arithmetic operations of addition, subtraction, multiplication, and division are computed by *functions of two inputs*.
- The addition function of f can be represented symbolically by $f(x,y) = x + y$, where $z = f(x,y)$.
  - The *independent variables* are x and y.
  - The *dependent variable* is z. The z output depends on the inputs x and y.

Here are Some Examples

For each function, evaluate the expression and interpret the result.

a) $f(5, -2)$ where $f(x,y) = xy$

b) $A(6,9)$, where $A(b,h) = \frac{1}{2}bh$ calculates the area of a triangle with a base of 6 inches and a height of 9 inches.

Solution

- $f(5, -2) = (5)(-2) = -10$.
- $A(6,9) = \frac{1}{2}(6)(9) = 27$

  If a triangle has a base of 6 inches and a height of 9 inches, the area of the triangle is 27 square inches.

Another Example

The equation $V = lwh$ is the volume of a rectangular box.

a) Solve $V = lwh$ for l.

b) Find l when $w = 6.5$ ft, $h = 9$ ft, and $V = 187.2$ ft^3 .

Solution

a)

$$V = lwh$$

$$\frac{V}{wh} = l$$

b)

$$\frac{V}{wh} = l$$

$$\frac{187.2}{(6.5)(9)} = l$$

$$3.2 \text{ ft} = l$$

What is a System of Linear Equations?

- A *linear equation in two variables* can be written in the form $ax + by = k$, where a, b, and k are constants, and a and b are not equal to 0.
- A pair of equations is called a *system of linear equations* because they involve solving more than one linear equation at once.
- A *solution* to a system of equations consists of an x-value and a y-value that satisfy both equations simultaneously.
- The set of all solutions is called the *solution set*.

How to Use the Method of Substitution to solve a system of two equations?

THE METHOD OF SUBSTITUTION

To use the method of substitution to solve a system of two equations in two variables, perform the following steps.

STEP 1: Choose a variable in one of the two equations. Solve the equation for that variable.

STEP 2: Substitute the result from **STEP 1** into the other equation and solve for the remaining variable.

STEP 3: Use the value of the variable from **STEP 2** to determine the value of the other variable. To do this, you may want to use the equation you found in **STEP 1**.

Note: To check your answer, substitute the value of each variable into the *given* equations. These values should satisfy *both* equations.

How to Solve the System Symbolically?

Solve the system symbolically.

$$4x + 2y = 8$$
$$3x - 7y = -11$$

Solution

Step 1: Solve one of the equations for one of the variables.

$$4x + 2y = 8$$
$$2y = -4x + 8$$
$$y = -2x + 4$$

Step 2: Substitute $-2x + 4$ for *y* in the second equation.

$$3x - 7(-2x + 4) = -11$$
$$3x + 14x - 28 = -11$$
$$x = 1$$

How to Solve the System Symbolically? (Cont.)

Step 3: Substitute *x* = 1 into the equation $y = -2x + 4$ from **Step 1**. We find that

$$y = 2.$$

Check:

- $3(1) - 7(2) = -11$ $\quad 4(1) + 2(2) = 8$
- The ordered pair is (1, 2) since the solutions check in *both* equations.

Example with Infinitely Many Solutions

- Solve the system. $\begin{aligned} 8x - 2y &= -4 \\ -4x + y &= 2 \end{aligned}$
- Solution
- Solve the second equation for *y*. $\begin{aligned} -4x + y &= 2 \\ y &= 4x + 2 \end{aligned}$
- Substitute 4*x* + 2 for *y* in the first equation, solving for *x*.

$$\begin{aligned} 8x - 2(4x + 2) &= -4 \\ 8x - 8x - 4 &= -4 \\ -4 &= -4 \end{aligned}$$

- The equation $-4 = -4$ is an identity that is always true and indicates that there are infinitely many solutions. The two equations are equivalent.

How to Solve the System Graphically and Symbolically?

The volume of a cylindrical container with a radius *r* and height *h* is computed by $V(r,h) = \pi r^2 h.$
The lateral surface area *S* of the container, excluding the circular top and bottom, is computed by

$$S(r,h) = 2\pi rh.$$

a) Write a system of equations whose solutions is the dimensions for the cylinder with a volume of 50 cubic centimeters and a lateral surface area of 130 square centimeters.
b) Solve the system of equations graphically and symbolically.

How to Solve the System Graphically and Symbolically? (Cont.)

Solution

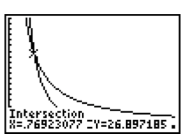
a) The equation *V*(*r*,*h*) = 50 and *S*(*r*,*h*) = 130 must be satisfied. This results in the following system of nonlinear equations.

$$\begin{aligned} \pi r^2 h &= 50 \\ 2\pi rh &= 130 \end{aligned}$$

b) Graphic Solution

$$h = \frac{50}{\pi r^2}$$

$$h = \frac{130}{2\pi r}$$



Let *r* correspond to *x* and *h* to *y*.

How to Solve the System Graphically and Symbolically?(Cont.)

Symbolic Solution

$$\frac{50}{\pi r^2} = \frac{130}{2\pi r}$$

$$2\pi r^2\left(\frac{50}{\pi r^2}\right) = 2\pi r^2\left(\frac{130}{2\pi r}\right)$$

$$100 = 130r$$

$$\frac{10}{13} = r$$

Because

$$r = \frac{10}{13} \approx 0.769,$$

$$h = \frac{130}{2\pi r} = \frac{130}{2\pi\left(10/13\right)} \approx 26.897$$

Next, Let's Look at Joint Variation

- A quantity may depend on more than one variable.

JOINT VARIATION

Let m and n be real numbers. Then z **varies jointly** as the mth power of x and the nth power of y if a nonzero real number k exists such that

$$z = kx^m y^n.$$

Example

- The area of a triangle varies jointly as the lengths of the base and the height. A triangle with base 20 inches and height of 8 inches has area 80 square inches. Find the area of a triangle with base 9 centimeters and height 12 centimeters.

Solution

Let A represent the area, b the base, and h the height of the triangle. Then $A = kbh$ for some number k.

Since $A = 80$ when b is 20 and h is 8,

$$80 = k(20)(8)$$

$$\frac{1}{2} = k$$

Example (Cont.)

The familiar formula for the area of a triangle is found.

$$A = \frac{1}{2}bh$$

So, when b = 9 centimeters and h = 12 centimeters,

$$A = \frac{1}{2}bh$$

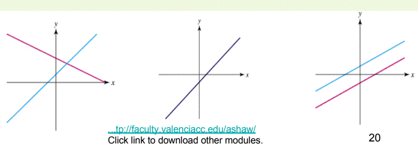
$$A = \frac{1}{2}(9)(12)$$

$$A = 54 \text{ square centimeters}$$

Possible Graphs of a System of Two Linear Equations in Two Variables

POSSIBLE GRAPHS OF A SYSTEM OF TWO LINEAR EQUATIONS IN TWO VARIABLES

1. The graphs of the two equations are distinct lines that intersect at one point. The system is *consistent*. There is one solution, which is given by the coordinates of the point of intersection. In this case the equations are *independent*.
2. The graphs of the two equations are the same line. The system is *consistent*. There are infinitely many solutions, and the equations are *dependent*.
3. The graphs of the two equations are distinct parallel lines. The system is *inconsistent*. There are no solutions.



How to Use Elimination Method to Solve System of Equations?

Use elimination to solve each system of equations, if possible. Identify the system as consistent or inconsistent. If the system is consistent, state whether the equations are dependent or independent. Support your results graphically.

a) $3x - y = 7$
$5x + y = 9$

b) $5x - y = 8$
$-5x + y = -8$

c) $x - y = 5$
$x - y = -2$

How to Use Elimination Method to Solve System of Equations? (Cont.)

Solution

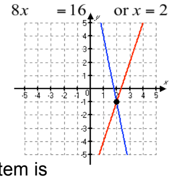
a) $3x - y = 7$
$5x + y = 9$

Eliminate y by adding the equations.

$$\begin{array}{r} 3x - y = 7 \\ \underline{5x + y = 9} \\ 8x \quad = 16 \end{array} \quad \text{or } x = 2$$

Find y by substituting $x = 2$ in either equation.

$$\begin{aligned} 3x - y &= 7 \\ 3(2) - y &= 7 \\ -y &= 1 \\ y &= -1 \end{aligned}$$



The solution is (2, −1). The system is consistent and the equations are independent.

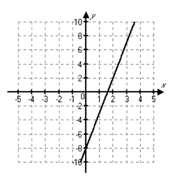
How to Use Elimination Method to Solve System of Equations? (Cont.)

b) $5x - y = 8$
$-5x + y = -8$

If we add the equations we obtain the following result.

$$\begin{array}{r} 5x - y = 8 \\ \underline{-5x + y = -8} \\ 0 = 0 \end{array}$$

The equation 0 = 0 is an identity that is always true.
The two equations are equivalent.
There are infinitely many solutions.
$\{(x, y) | \ 5x - y = 8\}$

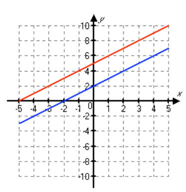


How to Use Elimination Method to Solve System of Equations? (Cont.)

c) $x - y = 5$
$x - y = -2$

If we subtract the second equation from the first, we obtain the following result.

$$\begin{array}{r} x - y = 5 \\ \underline{x - y = -2} \\ 0 = 7 \end{array}$$



The equation 0 = 7 is a contradiction that is never true. Therefore there are no solutions, and the system is inconsistent.

Let's Practice Using Elimination

Solve the system by using elimination.

$$3x - 4y = 1$$
$$2x + 3y = 12$$

Solution

Multiply the first equation by 3 and the second equation by 4. Addition eliminates the *y*-variable.

$$\begin{array}{l} 9x - 12y = 3 \\ \underline{8x + 12y = 48} \\ 17x \quad\ = 51 \quad \text{or } x = 3 \end{array}$$

Substituting $x = 3$ in $2x + 3y = 12$ results in

$2(3) + 3y = 12$ or $y = 2$

The solution is (3, 2).

One More Example

Solve the system of equations.

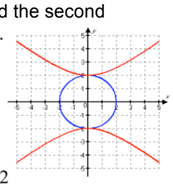
$$x^2 + y^2 = 4$$
$$2x^2 - 3y^2 = -12$$

Solution

Multiply the first equation by 3 and add the second equation, the variable y^2 is eliminated.

$$\begin{array}{l} 3x^2 + 3y^2 = 12 \\ \underline{2x^2 - 3y^2 = -12} \\ 5x^2 \quad\ = 0 \\ \qquad x = 0 \end{array}$$

$$\begin{array}{l} 0^2 + y^2 = 4 \\ \quad y^2 = 4 \\ \quad y = \pm 2 \end{array}$$



Then, substitute $x = 0$ to the first equation and solve for y.

How to Graph a System of Linear and Nonlinear Inequalities?

The graph of a linear inequality is a half-plane, which may include the boundary. The boundary line is included when the inequality includes a less than or equal to or greater than or equal to symbol.

To determine which part of the plane to shade, select a **test point**.

How to Graph a System of Linear and Nonlinear Inequalities? (Cont.)

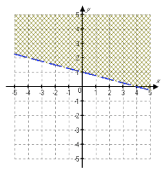
Graph the solution set to the inequality $x + 4y > 4$.

Solution

Graph the line $x + 4y = 4$ using a dashed line.

Use a test point to determine which half of the plane to shade.

Test Point	$x + 4y > 4$	True or False?
(4, 2)	$4 + 4(2) > 4$	True
(0, 0)	$0 + 4(0) > 4$	False



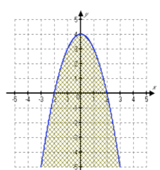
How to Graph a System of Linear and Nonlinear Inequalities? (Cont.)

Graph the solution set to the inequality $x^2 + y \leq 4$.

Solution

Graph the parabola determined by $y = 4 - x^2$ using a solid line.

Use a test point to determine which half of the plane to shade.



Test Point	$x^2 + y \leq 4$	True or False?
(0, 0)	$0^2 + 0 \leq 4$	True
(3, 1)	$3^2 + 1 \leq 4$	False

Example

Solve the system of inequalities by shading the solution set. Use the graph to identify one solution.

$x + y \leq 3$

$2x + y \geq 4$

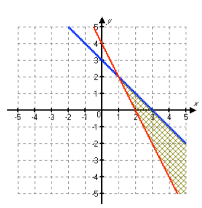
Solution

Solve each inequality for y.

$y \leq -x + 3$ (shade below line)

$y \geq -2x + 4$ (shade above line)

The point $(4, -2)$ is a solution.



What is Linear Programming?

Linear programming is a procedure used to optimize quantities such as cost and profit.

A linear programming problem consists of a linear **objective function** (an equation) and a system of linear inequalities called **constraints**. The solution set for the system of linear inequalities is called the set of **feasible solutions.**

If a solution exists, it occurs at a vertex in the region of feasible solutions.

Now, let's look at how to solve a linear programming problem.

How to Solve a Linear Programming Problem?

SOLVING A LINEAR PROGRAMMING PROBLEM

STEP 1: Read the problem carefully. Consider making a table to display the information given.

STEP 2: Use the table to write the objective function and all the constraints.

STEP 3: Sketch a graph of the region of feasible solutions. Identify all vertices or corner points.

STEP 4: Evaluate the objective function at each vertex. A maximum (or a minimum) occurs at a vertex. ***Note:*** If the region is unbounded, a maximum (or minimum) may not exist.

Example of Solving a Linear Programming Problem

Suppose a small company manufactures two products—VCR's and DVD players. Each VCR results in a $15 profit and each DVD player provides a profit of $50. Due to demand, the company must produce at least 10 and not more than 50 VCR's per day. The number of VCR's cannot exceed the number of DVD players, and the number of DVD players cannot exceed 60. How many of each type should the company manufacture to obtain the maximum profit?

Solution

Let x = VCR and y = DVD

The total daily profit P is given by $P = 15x + 50y$

Example of Solving a Linear Programming Problem (Cont.)

What else do we know?

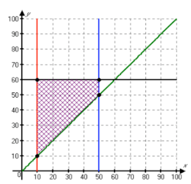
- The company produces from 10 to 50 VCR's per day, so the inequalities $x \geq 10$ and $x \leq 50$ must be satisfied.

- VCR's cannot exceed DVD players and the number of DVD players cannot exceed 60 indicate that $x \leq y$ and $y \leq 60$.

- The number of VCR's and DVD players cannot be negative, so $x \geq 0$ and $y \geq 0$.

Example of Solving a Linear Programming Problem (Cont.)

Graph the constraints. The shaded region is the set of feasible solutions.



Vertex	$P = 15x + 50y$
(10, 10)	15(10) + 50(10) =650
(10, 60)	15(10) + 50(60) = 3150
(50, 60)	15(50) + 50(60) = 3750
(50, 50)	15(50) + 50(50) = 3250

The maximum value of P is 3750 at vertex (50, 60). The maximum profit occurs when 50 VCR's and 60 DVD players are manufactured.

How to Solve a System of Linear Equations in Three Variables?

Solve the following system.

$$\begin{aligned} 3x + 9y + 6z &= -3 \\ 2x + y - z &= -2 \\ x + y + z &= 1 \end{aligned}$$

Solution

Step 1: Eliminate the variable z from equation one and two and then from equation two and three.

$$\begin{array}{rl} 3x + 9y + 6z = -3 & \text{Equation 1} \\ 12x + 6y - 6z = -12 & \text{Equation 2 times 6} \\ \hline 15x + 15y \quad = -15 & \text{Add} \end{array}
\qquad
\begin{array}{rl} 2x + y - z = -2 & \text{Equation 2} \\ x + y + z = 1 & \text{Equation 3} \\ \hline 3x + 2y \quad = -1 & \text{Add} \end{array}$$

How to Solve a System of Linear Equations in Three Variables? (Cont.)

Step 2: Take the two new equations and eliminate either variable.

$$\begin{aligned} 15x+15y&=-15 &\xrightarrow{\times 2}\quad 30x+30y&=-30 \\ 3x+\ \ 2y&=-1 &\xrightarrow{\times(-10)}\quad \underline{-30x-20y}&\underline{=10} \\ & & 10y&=-20 \\ & & y&=-2 \end{aligned}$$

Find *x* using *y* = –2.

$$\begin{aligned} 3x+2y&=-1 \\ 3x+2(-2)&=-1 \\ 3x&=3 \\ x&=1 \end{aligned}$$

How to Solve a System of Linear Equations in Three Variables? (Cont.)

Step 3: Substitute *x* = 1 and *y* = –2 in any of the given equations to find *z*.

$$\begin{aligned} x+y+z&=1 \\ 1+(-2)+z&=1 \\ -1+z&=1 \\ z&=2 \end{aligned}$$

The solution is (1, –2, 2).

Another Example

Eight hundred twenty zoo admissions were sold last Wednesday, which generated $4285 in revenue. The prices of the tickets were $4 for students, $7 for adults and $5 for seniors. There were 110 more student tickets sold than adults. Find the number of each type of ticket sold.

Solution

Let *x* = student tickets, *y* = adults and *z* = seniors

Write a system of three equations:

$$\begin{aligned} x+y+z&=820 \\ 4x+7y+5z&=4285 \\ x-y\qquad&=110 \end{aligned}$$

Another Example (Cont.)

Step 1: Eliminate *z* from equation 1 and 2. Multiply equation one by five and subtract equation 2.

$$\begin{array}{l} 5x+5y+5z=4100 \\ \underline{4x+7y+5z=4285} \\ x-2y \qquad = -185 \end{array}$$

Step 2: Use the third equation and the new equation to eliminate *x*. Subtract the equations.

$$\begin{array}{l} x-\ \ y=\ \ 110 \\ \underline{x-2y=-185} \\ \qquad y=\ \ 295 \end{array}$$

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Another Example (Cont.)

Step 2: Substitute *y* = 295 into the third equation to find *x*.

$$x-295=110$$
$$x=405$$

Step 3: Substitute *x* = 405, *y* = 295 into equation one to find *z*.

$$x+y+z=820$$
$$405+295+z=820$$
$$z=120$$

There were 405 students tickets, 295 adult tickets and 120 senior tickets sold.

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One More Example

Solve the system.

$$\begin{array}{r} 3x+5y-z=-2 \\ 4x-y+2z=1 \\ -6x-10y+2z=0 \end{array}$$

Solution

Step 1 Multiply equation one by 2 and add to equation two.

$$\begin{array}{l} 6x+10y-2z=-4 \\ \underline{4x-\ \ y+2z=\ \ 1} \\ 10x+9y \qquad =-3 \end{array}$$

Subtract equation three from equation two.

$$\begin{array}{l} 4x-\ \ y+2z=1 \\ \underline{-6x-10y+2z=1} \\ 10x+9y \qquad =0 \end{array}$$

Step 2 The two equations are inconsistent because the sum of 10*x* + 9*y* cannot be both −3 and 0.

Step 3 is not necessary the system of equations has no solution.

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What have we learned?

We have learned to

1. evaluate functions of two variables.
2. apply the method of substitution.
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4. solve problems involving joint variation.
5. recognize different types of linear systems.
6. apply the elimination method.
7. solve systems of linear and nonlinear inequalities.

What have we learned? (Cont.)

8. apply the method of substitution.
9. apply graphical and numerical methods to system of equations.
10. solve problems involving joint variation.
11. recognize different types of linear systems.
12. apply the elimination method.
13. solve systems of linear and nonlinear inequalities.

Credit

- Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:
- Rockswold, Gary, Precalculus with Modeling and Visualization, 3th Edition