STA 2023

Module 3A
Probability Concepts

Learning Objectives

Upon completing this module, you should be able to

1. Compute probabilities for experiments having equally likely outcomes.
2. Interpret probabilities, using the frequentist interpretation of probability.
3. State and understand the basic properties of probability.
4. Construct and interpret Venn diagrams.
5. Find and describe (not E), (A&B), and (A or B).
6. Determine whether two or more events are mutually exclusive.
7. Describe and use probability notation.
8. State and apply the special addition rule, the complementation rule, and the general addition rule.

What is a Random Phenomena?

- A random phenomenon is a situation in which we know what outcomes could happen, but we don't know which particular outcome did or will happen.

What is a Sample Space?

- In general, each occasion upon which we observe a random phenomenon is called a trial.
- At each trial, we note the value of the random phenomenon, and call it an outcome.
- When we combine outcomes, the resulting combination is an event.
- The collection of *all possible outcomes* is called the sample space.

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Sample Space and Event

Sample Space and Event

Sample space: The collection of all possible outcomes for an experiment.

Event: A collection of outcomes for the experiment, that is, any subset of the sample space.

The probability of an event reports the likelihood of the event's occurrence.

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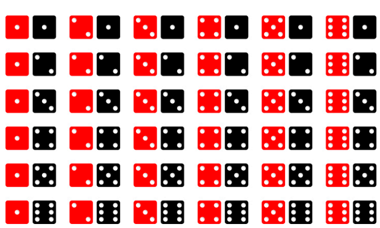
Possible Outcomes for Rolling a Pair of Dice



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Probability Notation

Remember, each occasion upon which we observe a random phenomenon is called a trial.

At each trial, we note the value of the random phenomenon, and call it an outcome.

When we combine outcomes, the resulting combination is an event.

The collection of *all possible outcomes* is called the sample space.

If E is an event, then $P(E)$ represents the probability that event E occurs. It is read "the probability of E."

Probability of an Event

Probability for Equally Likely Outcomes (f/N Rule)

Suppose an experiment has N possible outcomes, all equally likely. An event that can occur in f ways has probability f/N of occurring:

$$\text{Probability of an event} = \frac{f}{N}.$$

Number of ways event can occur (f)

Total number of possible outcomes (N)

In this case, we can represent the probability of an event as $P(E) = f/N$.

Basic Properties of Probabilities

Basic Properties of Probabilities

Property 1: The probability of an event is always between 0 and 1, inclusive.

Property 2: The probability of an event that cannot occur is 0. (An event that cannot occur is called an **impossible event.**)

Property 3: The probability of an event that must occur is 1. (An event that must occur is called a **certain event.**)

Again, the probability of an event reports the likelihood of the event's occurrence. We write $P(E)$ for the probability of the event E, and $0 \leq P(E) \leq 1$.

Trials are Independent

When thinking about what happens with combinations of outcomes, things are simplified if the individual trials are independent.

- Roughly speaking, this means that the outcome of one trial doesn't influence or change the outcome of another.
- For example, coin flips are independent.

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The Law of Large Numbers

- The Law of Large Numbers (LLN) says that the long-run *relative frequency* of repeated independent events gets closer and closer to a single value.
- We call the single value the probability of the event.
- Because this definition is based on repeatedly observing the event's outcome, this definition of probability is often called empirical probability.

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The Nonexistent Law of Averages

- The LLN says nothing about short-run behavior.
- Relative frequencies even out *only in the long run*, and this long run is *really* long (*infinitely* long, in fact).
- The so called Law of Averages (that an outcome of a random event that hasn't occurred in many trials is "due" to occur) doesn't exist at all.

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Modeling Probability

- When probability was first studied, a group of French mathematicians looked at games of chance in which all the possible outcomes were *equally likely*.
  - It's equally likely to get any one of six outcomes from the roll of a fair die.
  - It's equally likely to get heads or tails from the toss of a fair coin.
- However, keep in mind that events are *not* always equally likely.
  - A skilled basketball player has a better than 50-50 chance of making a free throw.

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Personal Probability

- In everyday speech, when we express a degree of uncertainty *without* basing it on long-run relative frequencies, we are stating subjective or personal probabilities.
- Personal probabilities don't display the kind of consistency that we will need probabilities to have, so we'll stick with formally defined probabilities.

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Three Rules of Working with Probability

We are dealing with probabilities now, not data, but the three rules don't change.

- Make a picture.
- Make a picture.
- Make a picture.

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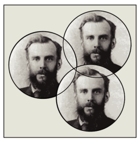
Three Rules of Working with Probability (cont.)

- The most common kind of picture to make is called a Venn diagram.



- We will see Venn diagrams in practice shortly…

Relationships Among Events

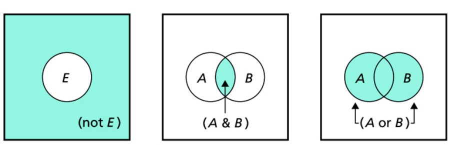
Relationships Among Events

(not *E*): The event "*E* does not occur"

(*A* & *B*): The event "both *A* and *B* occur"

(*A* or *B*): The event "either *A* or *B* or both occur"

Venn Diagram



(a) (b) (c)

(a) Event (not E) consists of all outcomes not in E.

(b) Event (A & B) consists of all outcomes when both event A and event B occur.

(c) Event (A or B) consists of all outcomes common to both event A and event B.

Mutually Exclusive Events

Mutually Exclusive Events

Two or more events are **mutually exclusive events** if no two of them have outcomes in common.

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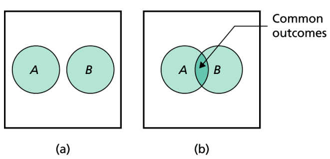
Mutually Exclusive Events



(a) Two mutually exclusive events.

(b) Two non-mutually exclusive events

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Formal Probability

1. Two requirements for a probability:
   - A probability is a number between 0 and 1.
   - For any event **A**, $0 \le P(\mathbf{A}) \le 1$.

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Formal Probability (cont.)

2. Probability Assignment Rule:
   - The probability of the set of all possible outcomes of a trial must be 1.
   - $P(\mathbf{S}) = 1$ (**S** represents the set of all possible outcomes.)

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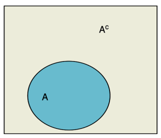
Formal Probability (cont.)

3. Complement Rule:
   - The set of outcomes that are *not* in the event **A** is called the complement of **A**, denoted $\mathbf{A^C}$.
   - The probability of an event occurring is 1 minus the probability that it doesn't occur: $P(\mathbf{A}) = 1 - P(\mathbf{A^C})$



The set **A** and its complement.

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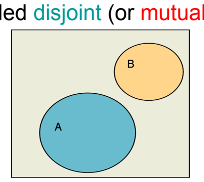
Formal Probability (cont.)

4. Special Addition Rule:
   - Events that have no outcomes in common (and, thus, cannot occur together) are called disjoint (or mutually exclusive).



Two disjoint sets, **A** and **B**.

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Formal Probability (cont.)

4. Special Addition Rule (cont.):
   - For two disjoint events **A** and **B**, the probability that one *or* the other occurs is the sum of the probabilities of the two events.
   - $P(\mathbf{A} \text{ or } \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B})$, provided that **A** and **B** are disjoint.

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Formal Probability (cont.)

5. Special Multiplication Rule (cont.):
   - For two independent events **A** and **B**, the probability that both **A** and **B** occur is the product of the probabilities of the two events.
   - $P(\mathbf{A} \text{ and } \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B})$, provided that **A** and **B** are independent.

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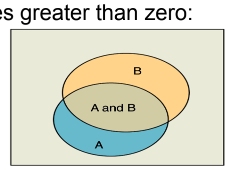
Formal Probability (cont.)

5. Special Multiplication Rule (cont.):
   - Two independent events **A** and **B** are not disjoint, provided the two events have probabilities greater than zero:



Two sets **A** and **B** that are not disjoint. The event (**A** and **B**) is their intersection.

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Formal Probability (cont.)

5. Special Multiplication Rule:
   - Many Statistics methods require an **Independence Assumption**, but *assuming* independence doesn't make it true.
   - Always *Think* about whether that assumption is reasonable before using the Special Multiplication Rule.

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Notation Alert

Sometimes, we use the notation
P(**A** or **B**) and *P*(**A** and **B**).

In other situations, you might see the following:

- *P*(**A** ∪ **B**) instead of *P*(**A** or **B**)
- *P*(**A** ∩ **B**) instead of *P*(**A** and **B**)

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The Complementation Rule

- In most situations where we want to find a probability, we'll often use the rules in combination.
- A good thing to remember is that sometimes it can be easier to work with the *complement* of the event we're really interested in.

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The General Addition Rule

- When two events **A** and **B** are disjoint, we can use the special addition rule for disjoint events:

  $P(\mathbf{A} \text{ or } \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B})$

- However, when our events are not disjoint, this earlier addition rule will double count the probability of *both* **A** and **B** occurring. Thus, we need the General Addition Rule.

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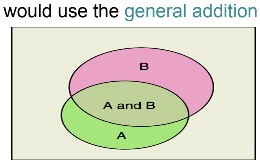
The General Addition Rule (cont.)

- General Addition Rule:
  - For any two events **A** and **B**,

    $P(\mathbf{A} \text{ or } \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \text{ and } \mathbf{B})$

- The following Venn diagram shows a situation in which we would use the general addition rule:



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Conditional Probability

- When we want the probability of an event from a *conditional* distribution, we write
  - $P(\mathbf{B}|\mathbf{A})$ and pronounce it "the probability of **B** *given* **A**."
- A probability that takes into account a given condition is called a conditional probability.

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Conditional Probability (Cont.)

- To find the probability of the event **B** *given* the event **A**, we restrict our attention to the outcomes in **A**. We then find in what fraction of *those* outcomes **B** also occurred.

$$P(\mathbf{B}|\mathbf{A}) = \frac{P(\text{A and B})}{P(\mathbf{A})}$$

- Note: $P(\mathbf{A})$ cannot equal 0, since we know that **A** has occurred.

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The General Multiplication Rule

- When two events **A** and **B** are independent, we can use the multiplication rule for independent events from Chapter 14:

  $P(\mathbf{A} \text{ and } \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B})$

- However, when our events are not independent, this earlier multiplication rule does not work. Thus, we need the General Multiplication Rule.

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The General Multiplication Rule (cont.)

- We encountered the general multiplication rule in the form of conditional probability.
- Rearranging the equation in the definition for conditional probability, we get the General Multiplication Rule:
  - For any two events **A** and **B**,

$$P(\mathbf{A} \text{ and } \mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B}|\mathbf{A})$$

or

$$P(\mathbf{A} \text{ and } \mathbf{B}) = P(\mathbf{B}) \times P(\mathbf{A}|\mathbf{B})$$

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Independence of Two Events

- Independence of two events means that the outcome of one event does not influence the probability of the other.
- With our new notation for conditional probabilities, we can now formalize this definition:
  - Events **A** and **B** are independent whenever $P(\mathbf{B}|\mathbf{A}) = P(\mathbf{B})$. (Equivalently, events **A** and **B** are independent whenever $P(\mathbf{A}|\mathbf{B}) = P(\mathbf{A})$.)

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Independent ≠ Disjoint

- Disjoint events *cannot* be independent! Well, why not?
  - Since we know that disjoint events have no outcomes in common, knowing that one occurred means the other didn't.
  - Thus, the probability of the second occurring changed based on our knowledge that the first occurred.
  - It follows, then, that the two events are *not* independent.
- A common error is to treat disjoint events as if they were independent, and apply the Multiplication Rule for independent events—don't make that mistake.

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Depending on Independence

- It's much easier to think about independent events than to deal with conditional probabilities.
  - It seems that most people's natural intuition for probabilities breaks down when it comes to conditional probabilities.
- Don't fall into this trap: whenever you see probabilities multiplied together, stop and ask whether you think they are really independent.

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Drawing Without Replacement

- Sampling without replacement means that once one object is drawn it doesn't go back into the pool.
  - We often sample without replacement, which doesn't matter too much when we are dealing with a large population.
  - However, when drawing from a small population, we need to take note and adjust probabilities accordingly.
- Drawing without replacement is just another instance of working with conditional probabilities.

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Tree Diagrams

- A tree diagram helps us think through conditional probabilities by showing sequences of events as paths that look like branches of a tree.
- Making a tree diagram for situations with conditional probabilities is consistent with our "make a picture" mantra.

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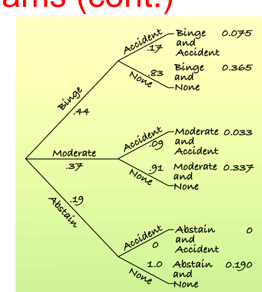
Tree Diagrams (cont.)

- Here is a nice example of a tree diagram and shows how we multiply the probabilities of the branches together:



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Let's Look at the Rules again

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The Special Addition Rule

The Special Addition Rule

If event A and event B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

More generally, if events $A, B, C, \ldots$ are mutually exclusive, then

$$P(A \text{ or } B \text{ or } C \text{ or } \cdots) = P(A) + P(B) + P(C) + \cdots.$$

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The Complementation Rule and The General Addition Rule

The Complementation Rule

For any event E,

$$P(E) = 1 - P(\text{not } E).$$

The General Addition Rule

If A and B are any two events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B).$$

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Conditional Probability and The Conditional Probability Rule

Conditional Probability

The probability that event B occurs given that event A occurs is called a *conditional probability.* It is denoted $\boldsymbol{P(B \mid A)}$, which is read "the probability of B given A." We call A the **given event.**

The Conditional Probability Rule

If A and B are any two events with $P(A) > 0$, then

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)}.$$

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The General Multiplication Rule

The General Multiplication Rule

If A and B are any two events, then

$$P(A \,\&\, B) = P(A) \cdot P(B \mid A).$$

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Independent Events and The Special Multiplication Rule

Independent Events

Event B is said to be **independent** of event A if $P(B \mid A) = P(B)$.

In general, two events are independent if knowing whether one event occurs does not alter the probability that the other event occurs.

The Special Multiplication Rule

If events $A, B, C, \ldots$ are independent, then

$$P(A \,\&\, B \,\&\, C \,\&\, \cdots) = P(A) \cdot P(B) \cdot P(C) \cdots.$$

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Factorials and The Permutations Rule

Factorials

The product of the first k positive integers (counting numbers) is called **k factorial** and is denoted $k!$. In symbols,

$$k! = k(k-1)\cdots 2 \cdot 1.$$

We also define $0! = 1$.

The Permutations Rule

The number of possible permutations of r objects from a collection of m objects is given by the formula

$$_mP_r = \frac{m!}{(m-r)!}.$$

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What Can Go Wrong?

Beware of probabilities that don't add up to 1.

- To be a legitimate probability assignment, the sum of the probabilities for all possible outcomes must total 1.

Don't add probabilities of events if they're not disjoint.

- Events must be disjoint to use the Addition Rule.

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What Can Go Wrong? (cont.)

- Don't multiply probabilities of events if they're not independent.
  - The multiplication of probabilities of events that are not independent is one of the most common errors people make in dealing with probabilities.

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What Can Go Wrong? (Cont.)

- Don't use a simple probability rule where a general rule is appropriate:
  - Don't assume that two events are independent or disjoint without checking that they are.
- Don't find probabilities for samples drawn without replacement as if they had been drawn with replacement.
- Don't confuse "disjoint" with "independent."

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What have we learned?

We have learned to:

1. Compute probabilities for experiments having equally likely outcomes.
2. Interpret probabilities, using the frequentist interpretation of probability.
3. State and understand the basic properties of probability.
4. Construct and interpret Venn diagrams.
5. Find and describe (not E), (A&B), and (A or B).
6. Determine whether two or more events are mutually exclusive.
7. Describe and use probability notation.
8. State and apply the special addition rule, the complementation rule, and the general addition rule

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Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbooks.

- Weiss, Neil A., Introductory Statistics, 8th Edition
- Bock, David E., Stats: Data and Models, 3rd Edition

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