MAC 1114

Module 10

Polar Form of Complex Numbers

Learning Objectives

Upon completing this module, you should be able to:

1. Identify and simplify imaginary and complex numbers.
2. Add and subtract complex numbers.
3. Simplify powers of *i*.
4. Multiply complex numbers.
5. Use property of complex conjugates.
6. Divide complex numbers.
7. Solve quadratic equations for complex solutions.
8. Convert between rectangular form and trigonometric (polar) form.

Polar Form of Complex Numbers

There are two major topics in this module:

- Complex Numbers
- Trigonometric (Polar) Form of Complex Numbers

Quick Review on Complex Numbers

- $i^2 = -1$ and $i = \sqrt{-1}$
- *i* is the imaginary unit
- Numbers in the form *a + bi* are called complex numbers
  - *a* is the real part
  - *b* is the imaginary part

If $a > 0$, then $\sqrt{-a} = i\sqrt{a}$.

Examples

- a) $\sqrt{-25} = i\sqrt{25} = 5i$ b) $\sqrt{-30} = i\sqrt{30}$
- c) $\sqrt{-125} = i\sqrt{125} = i\sqrt{25 \cdot 5} = 5i\sqrt{5}$
- d) $\sqrt{-3} \cdot \sqrt{-3} = i\sqrt{3} \cdot i\sqrt{3}$
  $= i^2\left(\sqrt{3}\right)^2$
  $= -1 \cdot 3$
  $= -3$

  e) $\dfrac{\sqrt{-98}}{\sqrt{49}} = \dfrac{-\sqrt{98}}{\sqrt{49}}$
  $= i\sqrt{\dfrac{98}{49}} = i\sqrt{2}$

Example of Solving Quadratic Equations

- Solve *x* = –25
- Take the square root on both sides.

$$x^2 = -25$$
$$\sqrt{x^2} = \pm\sqrt{-25}$$
$$x = \pm i\sqrt{25}$$
$$x = \pm 5i$$

- The solution set is {±5*i*}.

Another Example

- Solve: $x^2 + 54 = 0$

$$x^2 + 54 = 0$$
$$x^2 = -54$$
$$x = \pm\sqrt{-54}$$
$$x = \pm i\sqrt{54} = \pm i\sqrt{9 \times 6}$$
$$x = \pm 3i\sqrt{6}$$

- The solution set is $\left\{\pm 3i\sqrt{6}\right\}$.

Multiply and Divide

- Multiply: $\sqrt{-8} \cdot \sqrt{-8}$.

$$\sqrt{-8} \cdot \sqrt{-8} = i\sqrt{8} \times i\sqrt{8}$$
$$= i^2 \times \left(\sqrt{8}\right)^2 \qquad i^2 = -1$$
$$= -1 \times 8$$
$$= -8$$

- Divide: $\dfrac{\sqrt{-56}}{\sqrt{8}}$.

$$\frac{\sqrt{-56}}{\sqrt{8}} = \frac{i\sqrt{56}}{\sqrt{8}}$$
$$= i\sqrt{\frac{56}{8}}$$
$$= i\sqrt{7}$$

Addition and Subtraction of Complex Numbers

- For complex numbers $a + bi$ and $c + di$,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

- **Examples**

$(4 - 6i) + (-3 + 7i)$	$(10 - 4i) - (5 - 2i)$
$= [4 + (-3)] + [-6 + 7]i$	$= (10 - 5) + [-4 - (-2)]i$
$= 1 + i$	$= 5 - 2i$

Multiplication of Complex Numbers

- For complex numbers $a + bi$ and $c + di$,

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

- The product of two complex numbers is found by multiplying as if the numbers were binomials and using the fact that $i^2 = -1$.

Let's Practice Some Multiplication of Complex Numbers

- $(2 - 4i)(3 + 5i)$

$$\begin{aligned} &= 2(3) + 2(5i) - 4i(3) - 4i(5i) \\ &= 6 + 10i - 12i - 20i^2 \\ &= 6 - 2i - 20(-1) \\ &= 26 - 2i \end{aligned}$$

- $(7 + 3i)^2$

$$\begin{aligned} &= 7^2 + 2(7)(3i) + (3i)^2 \\ &= 49 + 42i + 9i^2 \\ &= 49 + 42i + 9(-1) \\ &= 40 + 42i \end{aligned}$$

Powers of i

- $i^1 = i$ $\quad i^5 = i$ $\quad i^9 = i$
- $i^2 = -1$ $\quad i^6 = -1$ $\quad i^{10} = -1$
- $i^3 = -i$ $\quad i^7 = -i$ $\quad i^{11} = -i$
- $i^4 = 1$ $\quad i^8 = 1$ $\quad i^{12} = 1$

and so on.

Simplifying Examples

- i^{17}

Since $i^4 = 1$,

$i^{17} = (i^4)^4 \cdot i$
$= 1 \cdot i$
$= i$

- i^{-4}

$$\frac{1}{i^4} = \frac{1}{1} = 1$$

Properties of Complex Conjugates

For real numbers a and b,
$(a + bi)(a - bi) = a^2 + b^2$.

The product of a complex number and its conjugate is always a real number.

- **Example**

$$\frac{5+3i}{2-i}$$
$$= \frac{(5+3i)(2+i)}{(2-i)(2+i)}$$
$$= \frac{10+5i+6i+3i^2}{4-i^2}$$
$$= \frac{7+11i}{5}$$
$$= \frac{7}{5} + \frac{11i}{5}$$

Complex Plane

- We modify the familiar coordinate system by calling the horizontal axis the **real axis** and the vertical axis the **imaginary axis**.
- Each complex number $a + bi$ determines a unique position vector with initial point (0, 0) and terminal point (a, b).

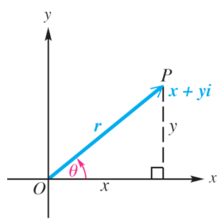
Relationships Among *z*, *y*, *r*, and θ

- $x = r\cos\theta$

  $y = r\sin\theta$

  $r = \sqrt{x^2 + y^2}$

  $\tan\theta = \dfrac{y}{x}, \quad \text{if } x \neq 0$



Trigonometric (Polar) Form of a Complex Number

- The expression $r(\cos\theta + i\sin\theta)$

  is called the **trigonometric form** or (**polar form**) of the complex number *x* + *yi.* The expression cos θ + *i* sin θ is sometimes abbreviated cis θ.

  Using this notation

  $$r(\cos\theta + i\sin\theta) \text{ is written } r \text{ cis } \theta.$$

Example

- Express 2(cos 120° + *i* sin 120°) in rectangular form.
- $\cos 120^\circ = -\dfrac{1}{2}$

  $\sin 120^\circ = \dfrac{\sqrt{3}}{2}$

  $$2(\cos 120^\circ + i\sin 120^\circ) = 2\left(-\frac{1}{2}, i\frac{\sqrt{3}}{2}\right)$$
  $$= -1 + i\sqrt{3}$$
- Notice that the real part is negative and the imaginary part is positive, this is consistent with 120 degrees being a quadrant II angle.

How to Convert from Rectangular Form to Polar Form?

- Step 1 Sketch a graph of the number $x + yi$ in the complex plane.
- Step 2 Find r by using the equation $r = \sqrt{x^2 + y^2}$.
- Step 3 Find θ by using the equation $\tan\theta = \frac{y}{x}, x \neq 0$ choosing the quadrant indicated in Step 1.

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Example

- Example: Find trigonometric notation for $-1 - i$.
- First, find r.

$$r = \sqrt{a^2 + b^2} \qquad \sin\theta = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2} \qquad \cos\theta = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$r = \sqrt{(-1)^2 + (-1)^2} \qquad \theta = \frac{5\pi}{4}$$

$$r = \sqrt{2}$$

- Thus, $-1 - i = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ or $\sqrt{2}\text{ cis }\frac{5\pi}{4}$

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What have we learned?

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Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Margaret L. Lial, John Hornsby, David I. Schneider, Trigonometry, 8th Edition