MAC 1114

Module 12
Polar and Parametric Equations

Learning Objectives

Upon completing this module, you should be able to:

1. Use the polar coordinate system.
2. Graph polar equations.
3. Solve polar equations.
4. Convert between a polar equation and a rectangular equation.
5. Graph a plane curve defined parametrically.
6. Find an equivalent rectangular equation for the plane curve.
7. Use parametric equations to solve applications.

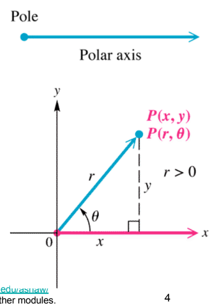
Polar and Parametric Equations

There are two major topics in this module:

- Polar Equations and Graphs
- Parametric Equations, Graphs, and Applications

Polar Coordinate System

- The **polar coordinate system** is based on a point, called the **pole**, and a ray, called the **polar axis**.



Rectangular and Polar Coordinates

- If a point has rectangular coordinates (x, y) and polar coordinates (r, θ), then these coordinates are related as follows.

$$x = r\cos\theta \qquad y = r\sin\theta$$

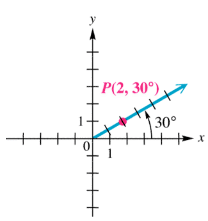
$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}, x \neq 0$$

Example

- Plot the point on a polar coordinate system. Then determine the rectangular coordinates of the point.

$P(2, 30°)$

$r = 2$ and $\theta = 30°$, so point P is located 2 units from the origin in the positive direction making a 30° angle with the polar axis.



Example (Cont.)

- Using the conversion formulas:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x = 2\cos 30^\circ \qquad y = 2\sin 30^\circ$$

$$x = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \qquad y = 2\left(\frac{1}{2}\right) = 1$$

- The rectangular coordinates are $\left(\sqrt{3}, \frac{1}{2}\right)$.

Another Example

- Convert (4, 2) to polar coordinates.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{4^2 + 2^2}$$
$$r = \sqrt{16 + 4}$$
$$r = \sqrt{20} = 2\sqrt{5}$$

$$\theta = \frac{2}{4} = \frac{1}{2}$$
$$\theta \approx 26.6^{o}$$

- Thus (r, θ) = $\left(2\sqrt{5}, 26.6^\circ\right)$

How to Convert Between Rectangular and Polar Equations?

- To convert a rectangular equation into a polar equation, use

$x = r\cos\theta$ and $y = r\sin\theta$

and solve for r.

For the linear equation $ax + by = c$,

you will get the polar equation $r = \dfrac{c}{a\cos\theta + b\sin\theta}$.

Example of Converting a Rectangular Equation into a Polar Equation

- Convert $x + 2y = 10$ into a polar equation.

$$x + 2y = 10$$

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

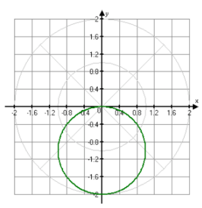
$$r = \frac{10}{1\cos\theta + 2\sin\theta}$$

$$r = \frac{10}{\cos\theta + 2\sin\theta}$$

How to Graph a Polar Equation?

- Graph $r = -2 \sin \theta$

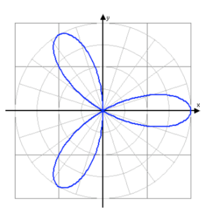
θ	*r*	θ	*r*
0	0	135	-1.414
30	-1	150	-1
45	-1.414	180	0
60	-1.732	270	2
90	-2	315	1.414
120	-1.732	330	1



Example of Graphing a Polar Equation

- Graph $r = 2 \cos 3\theta$

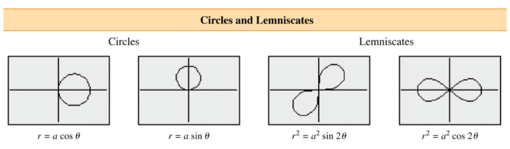
θ	0	15	30	45	60	75	90
r	2	1.41	0	−1.41	−2	−1.41	0



How to Convert a Polar Equation into a Rectangular Equation?

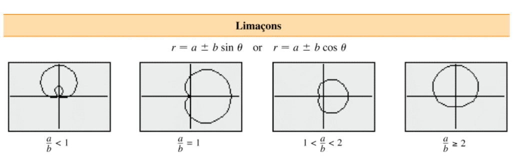
- Convert $r = -3 \cos \theta - \sin \theta$ into a rectangular equation.
- $$\begin{aligned} r &= -3\cos\theta - \sin\theta \\ r^2 &= -3r\cos\theta - r\sin\theta \\ x^2 + y^2 &= -3x - y \\ x^2 &= -3x - y - y^2 \end{aligned}$$

Circles and Lemniscates



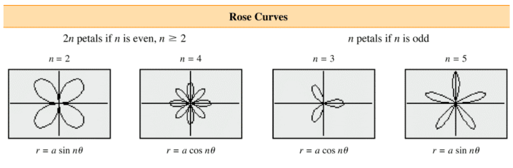
What are the primary differences between circles and lemniscates?

Limacons



Note: The curve with heart shape above is called cardioid. Cardioids are a special case of limacons. How many cardioids are here?

Rose Curves



What do you notice about the number of petals when n is even and when n is odd?

Parametric Equations of a Plane Curve

- A **plane curve** is a set of points (x, y) such that $x = f(t)$, $y = g(t)$, and f and g are both defined on an interval I. The equations $x = f(t)$ and $y = g(t)$ are **parametric equations** with **parameter t**.

How to Graph a Plane Curve Defined Parametrically?

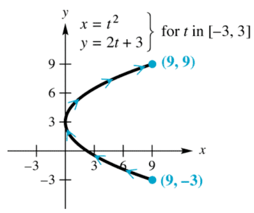
Example: Let $x = t^2$ and $y = 2t + 3$, for t in $[-3,3]$. Graph the set of ordered pairs (x, y).

Solution: Make a table of corresponding values of t, x, and y over the domain of t.

t	x	y
−3	9	−3
−2	4	−1
−1	1	1
0	0	3
1	1	5
2	4	7
3	9	9

How to Graph a Plane Curve Defined Parametrically? (Cont.)

Plotting the points shows a graph of a portion of a parabola with horizontal axis $y = 3$. The arrowheads indicate the direction the curve traces as *t* increases.



How to Find an Equivalent Rectangular Equation?

Example: Find a rectangular equation for the plane curve of the previous example defined as follows.
$x = t^2$, $y = 2t + 3$, for t in $[-3, 3]$

Solution: Solve either equation for *t*.

$$y = 2t + 3$$
$$2t = y - 3$$
$$t = \frac{y-3}{2}$$

How to Find an Equivalent Rectangular Equation? (Cont.)

Now substitute this result into the first equation to get

$$x = t^2 = \left(\frac{y-3}{2}\right)^2 = \frac{(y-3)^2}{4} \quad \text{or} \quad 4x = (y-3)^2.$$

- This is the equation of a horizontal parabola opening to the right. Because t is in $[-3, 3]$, x is in $[0, 9]$ and y is in $[-3, 9]$. This rectangular equation must be given with its restricted domain as $4x = (y-3)^2$, for x in $[0, 9]$.

How to Graph a Plane Curve Defined Parametrically?

Graph the plane curve defined by $x = 2\sin(t)$,
$y = 3\cos(t)$, for t in $[0,2\pi]$.

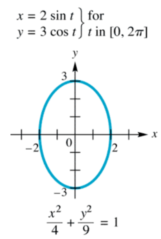
Use the fact that $\sin^2(t) + \cos^2(t) = 1$. Square both sides of each equation; solve one for $\sin^2(t)$, the other for $\cos^2(t)$.

$$x = 2\sin t \qquad\qquad y = 3\cos t$$
$$x^2 = 4\sin^2 t \qquad\qquad y^2 = 9\cos^2 t$$
$$\frac{x^2}{4} = \sin^2 t \qquad\qquad \frac{y^2}{9} = \cos^2 t$$

How to Graph a Plane Curve Defined Parametrically? (Cont.)

Now add corresponding sides of the two equations.

$$\frac{x^2}{4} + \frac{y^2}{9} = \sin^2 t + \cos^2 t$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$



This is the equation of an *ellipse*.

How to Find Alternative Parametric Equation Forms?

Give two parametric representations for the equation of the parabola $y = (x + 5)^2 + 3$.

Solution:

The simplest choice is to let

$x = t$, $\quad y = (t + 5)^2 + 3 \quad$ for t in $(-\infty, \infty)$

Another choice, which leads to a simpler equation for y, is

$x = t + 5$, $\quad y = t^2 + 3 \quad$ for t in $(-\infty, \infty)$.

Application

A small rocket is launched from a table that is 3.36 ft above the ground. Its initial velocity is 64 ft per sec, and it is launched at an angle of 30° with respect to the ground. Find the rectangular equation that models its path. What type of path does the rocket follow?

Solution: The path of the rocket is defined by the parametric equations

$x = (64 \cos 30°)t$ and $y = (64 \sin 30°)t - 16t^2 + 3.36$

Or equivalently,

$$x = 32\sqrt{3}t \quad \text{and} \quad y = -16t^2 + 32t + 3.36.$$

Application (Cont.)

From $x = 32\sqrt{3}t$, we obtain $t = \frac{x}{32\sqrt{3}}$.

Substituting into the other parametric equations for t yields

$$y = -16\left(\frac{x}{32\sqrt{3}}\right)^2 + 32\left(\frac{x}{32\sqrt{3}}\right) + 3.36.$$

Simplifying, we find that the rectangular equation is

$$y = -\frac{1}{192}x^2 + \frac{\sqrt{3}}{3}x + 3.36.$$

Because the equation defines a parabola, the rocket follows a parabolic path.

What have we learned?

We have learned to:

1. Use the polar coordinate system.
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Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Margaret L. Lial, John Hornsby, David I. Schneider, Trigonometry, 8th Edition

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