MAC 1114

Module 5

Trigonometric Identities I

Learning Objectives

Upon completing this module, you should be able to:

1. Recognize the fundamental identities: reciprocal identities, quotient identities, Pythagorean identities and negative-angle identities.
2. Express the fundamental identities in alternate forms.
3. Use the fundamental identities to find the values of other trigonometric functions from the value of a given trigonometric function.
4. Express any trigonometric functions of a number or angle in terms of any other functions.
5. Simplify trigonometric expressions using the fundamental identities.
6. Use fundamental identities to verify that a trigonometric equation is an identity.
7. Apply the sum and difference identities for cosine.

Trigonometric Identities

There are three major topics in this module:

- Fundamental Identities
- Verifying Trigonometric Identities
- Sum and Difference Identities for Cosine

Fundamental Identities

- Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

- Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Tip: Memorize these Identities.

Fundamental Identities (Cont.)

- Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

- Negative-Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Tip: Memorize these Identities.

Example: If $\tan\theta = -\frac{5}{3}$ and θ is in quadrant II, find each function value.

- a) sec (θ)

To find the value of this function, look for an identity that relates tangent and secant.

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$
$$\left(-\frac{5}{3}\right)^2 + 1 = \sec^2\theta$$
$$\frac{25}{9} + 1 = \sec^2\theta$$
$$\frac{34}{9} = \sec^2\theta$$
$$\sec\theta = -\sqrt{\frac{34}{9}}$$
$$\sec\theta = -\frac{\sqrt{34}}{3}$$

Tip: Use Pythagorean Identities.

Example: If $\tan\theta = -\frac{5}{3}$ and θ is in quadrant II, find each function value. (Cont.)

- b) sin (θ)

Tip: Use Quotient Identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cos\theta\tan\theta = \sin\theta$$
$$\left(\frac{1}{\sec\theta}\right)\tan\theta = \sin\theta$$
$$\left(-\frac{3\sqrt{34}}{34}\right)\left(-\frac{5}{3}\right) = \sin\theta$$
$$\frac{5\sqrt{34}}{34} = \sin\theta$$

- c) cot (−θ)

Tip: Use Reciprocal and Negative-Angle Identities.

$$\cot(-\theta) = \frac{1}{\tan(-\theta)}$$
$$\cot(-\theta) = \frac{1}{-\tan\theta}$$
$$\cot(-\theta) = \frac{1}{-\left(-\frac{5}{3}\right)} = \frac{3}{5}$$

Example of Expressing One Function in Terms of Another

- Express cot(*x*) in terms of sin(*x*).

- Tip: Use Pythagorean Identities.

$$1+\cot^2 x = \csc^2 x$$
$$\frac{1}{1+\cot^2 x} = \frac{1}{\csc^2 x}$$
$$\frac{1}{1+\cot^2 x} = \sin^2 x$$
$$\pm\sqrt{\frac{1}{1+\cot^2 x}} = \sqrt{\sin^2 x}$$
$$\sin x = \frac{\pm 1}{\sqrt{1+\cot^2 x}}$$
$$\sin x = \frac{\pm\sqrt{1+\cot^2 x}}{1+\cot^2 x}$$

Example of Rewriting an Expression in Terms of Sine and Cosine

- Rewrite cot θ − tan θ in terms of sin θ and cos θ .

Tip: Use Quotient Identities.

- $$\cot\theta - \tan\theta = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\sin\theta\cos\theta}$$

Hints for Verifying Identities

- 1. Learn the fundamental identities given in the last section. Whenever you see either side of a fundamental identity, the other side should come to mind. Also, be aware of equivalent forms of the fundamental identities. For example $\sin^2\theta = 1 - \cos^2\theta$ is an alternative form of the identity $\sin^2\theta + \cos^2\theta = 1.$
- 2. Try to rewrite the more complicated side of the equation so that it is identical to the simpler side.

Hints for Verifying Identities (Cont.)

- 3. It is sometimes helpful to express all trigonometric functions in the equation in terms of sine and cosine and then simplify the result.
- 4. Usually, any factoring or indicated algebraic operations should be performed. For example, the expression $\sin^2 x + 2\sin x + 1$ can be factored as $(\sin x + 1)^2.$ The sum or difference of two trigonometric expressions such as $\frac{1}{\sin\theta} + \frac{1}{\cos\theta},$ can be added or subtracted in the same way as any other rational expression.

Hints for Verifying Identities (Cont.)

- 5. As you select substitutions, keep in mind the side you are changing, because it represents your goal. For example, to verify the identity

$$\tan^2 x + 1 = \frac{1}{\cos^2 x}$$

try to think of an identity that relates tan *x* to cos *x*. In this case, since $\sec x = \frac{1}{\cos x}$ and $\sec^2 x = \tan^2 x + 1,$ the secant function is the best link between the two sides.

Hints for Verifying Identities (Cont.)

- 6. If an expression contains 1 + sin x, multiplying both the numerator and denominator by 1 − sin x would give 1 − $\sin^2 x$, which could be replaced with $\cos^2 x$. Similar results for 1 − sin x, 1 + cos x, and 1 − cos x may be useful.
- Remember that verifying identities is NOT the same as solving equations.

Example of Verifying an Identity: Working with One Side

- Prove the identity $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$
- Solution: Start with the left side.

$$(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$$
$$\left(\frac{\sin^2 x}{\cos^2 x} + 1\right)(\cos^2 x - 1) = -\tan^2 x$$
$$\sin^2 x - \frac{\sin^2 x}{\cos^2 x} + \cos^2 x - 1 = -\tan^2 x$$
$$\sin^2 x + \cos^2 x - \frac{\sin^2 x}{\cos^2 x} - 1 = -\tan^2 x$$
$$1 - \frac{\sin^2 x}{\cos^2 x} - 1 = -\tan^2 x$$
$$-\frac{\sin^2 x}{\cos^2 x} = -\tan^2 x$$
$$-\tan^2 x = -\tan^2 x$$

Example of Verifying an Identity: Working with One Side

- Prove the identity

$$\frac{1}{\sec x \tan x} = \csc x - \sin x$$

- Solution—start with the right side

$$\frac{1}{\sec x \tan x} = \csc x - \sin x$$
$$= \frac{1}{\sin x} - \sin x$$
$$= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}$$

- continued

$$\frac{1}{\sec x \tan x} = \frac{1 - \sin^2 x}{\sin x}$$
$$= \frac{\cos^2 x}{\sin x}$$
$$= \frac{\cos x}{\sin x} \cdot \frac{\cos x}{1}$$
$$= \cot x \cdot \cos x$$
$$= \frac{1}{\tan x} \cdot \frac{1}{\sec x}$$
$$\frac{1}{\sec x \tan x} = \frac{1}{\sec x \tan x}$$

Example of Verifying an Identity: Working with One Side

- Prove the identity $\frac{\tan x + \cot y}{\tan x \cot y} = \tan y + \cot x$
- Start with the left side.

$$\frac{\tan x + \cot y}{\tan x \cot y} = \tan y + \cot x$$
$$\frac{\tan x}{\tan x \cot y} + \frac{\cot y}{\tan x \cot y} =$$
$$\frac{1}{\cot y} + \frac{1}{\tan x} =$$
$$\tan y + \cot x = \tan y + \cot x$$

Example of Verifying an Identity: Working with Both Sides

- Verify that the following equation is an identity.

$$\frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha} = \frac{1 + 2\sin\alpha + \sin^2\alpha}{\cos^2\alpha}$$

- Solution: Since both sides appear complex, verify the identity by changing each side into a common third expression.

Example of Verifying an Identity: Working with Both Sides

- Left side: Multiply numerator and denominator by $\cos\alpha$.

$$\frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha} = \frac{(\sec\alpha + \tan\alpha)\cos\alpha}{(\sec\alpha - \tan\alpha)\cos\alpha}$$
$$= \frac{\sec\alpha\cos\alpha + \tan\alpha\cos\alpha}{\sec\alpha\cos\alpha - \tan\alpha\cos\alpha}$$
$$= \frac{1 + \tan\alpha\cos\alpha}{1 - \tan\alpha\cos\alpha}$$
$$= \frac{1 + \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha}$$
$$= \frac{1 + \sin\alpha}{1 - \sin\alpha}$$

Example of Verifying an Identity: Working with Both Sides Continued

- Right Side: Begin by factoring.

$$\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}=\frac{(1+\sin\alpha)^2}{\cos^2\alpha}$$
$$=\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}$$
$$=\frac{(1+\sin\alpha)^2}{(1+\sin\alpha)(1-\sin\alpha)}$$
$$=\frac{1+\sin\alpha}{1-\sin\alpha}$$

- We have shown that $\frac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha}=\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}$

verifying that the given equation is an identity.

Cosine of a Sum or Difference

- $\cos(A-B)=\cos A\cos B+\sin A\sin B$
  $\cos(A+B)=\cos A\cos B-\sin A\sin B$
  Tip: Memorize these Sum and Difference Identities for cosine.
- Find the *exact* value of cos 15°.
- $\cos 15^\circ=\cos(45^\circ-30^\circ)$

$$=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$$
$$=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2}$$
$$=\frac{\sqrt{6}+\sqrt{2}}{4}$$

Example

Tip: Apply cosine difference identity

- $\cos\frac{5\pi}{12}$

$$=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

- $\cos 87^\circ\cos 93^\circ-\sin 87^\circ\sin 93^\circ$

$$=\cos(87^\circ+93^\circ)$$
$$=\cos 180^\circ$$
$$=-1$$

Tip: Memorize these Sum and Difference Identities.

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

Example: Reducing

- Write cos (180° − θ) as a trigonometric function of θ.

  Tip: Apply cosine difference identity here.

- $$\begin{aligned}\cos(180^\circ - \theta) &= \cos 180^\circ \cos\theta + \sin 180^\circ \sin\theta \\ &= (-1)\cos\theta + (0)\sin\theta \\ &= -\cos\theta\end{aligned}$$

Cofunction Identities

- $$\begin{aligned}\cos(90^\circ - \theta) &= \sin\theta & \cot(90^\circ - \theta) &= \tan\theta \\ \sin(90^\circ - \theta) &= \cos\theta & \sec(90^\circ - \theta) &= \csc\theta \\ \tan(90^\circ - \theta) &= \cot\theta & \csc(90^\circ - \theta) &= \sec\theta\end{aligned}$$

- Similar identities can be obtained for a real number domain by replacing 90° with π/2.

Example: Using Cofunction Identities

- Find an angle that satisfies sin (−30°) = cos θ
- $$\begin{aligned}\sin(-30^\circ) &= \cos\theta \\ \sin(-30^\circ) &= \sin(90^\circ - \theta) \\ -30^\circ &= 90^\circ - \theta \\ \theta &= 120^\circ\end{aligned}$$

What have we learned?

We have learned to:

1. Recognize the fundamental identities: reciprocal identities, quotient identities, Pythagorean identities and negative-angle identities.
2. Express the fundamental identities in alternate forms.
3. Use the fundamental identities to find the values of other trigonometric functions from the value of a given trigonometric function.
4. Express any trigonometric functions of a number or angle in terms of any other functions.
5. Simplify trigonometric expressions using the fundamental identities.
6. Use fundamental identities to verify that a trigonometric equation is an identity.
7. Apply the sum and difference identities for cosine.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Margaret L. Lial, John Hornsby, David I. Schneider, Trigonometry, 8th Edition