MAC 1114

Module 7

Inverse Circular Functions and Trigonometric Equations

Learning Objectives

Upon completing this module, you should be able to:

1. Identify inverse functions.
2. Define and use the inverse sine function.
3. Define and use the inverse cosine function.
4. Define and use the inverse tangent function.
5. Find inverse function values.
6. Determine the solutions of a trigonometric equation within a given interval.
7. Use the inverse trigonometric functions to solve trigonometric equations.
8. Solve trigonometric equations that arise from applied problems.

Inverse Circular Functions and Trigonometric Equations

There are two major topics in this module:

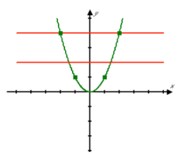
- Inverse Circular Functions
- Trigonometric Equations

A Quick Review on Inverse Function

- In a one-to-one function, each *x*-value corresponds to only one *y*-value and each *y*-value corresponds to only one *x*-value.
- If a function *f* is one-to-one, then *f* has an inverse function f^{-1}.
- The domain of *f* is the range of f^{-1} and the range of *f* is the domain of f^{-1}.
- The graphs of *f* and f^{-1} are reflections of each other about the line $y = x$.
- To find $f^{-1}(x)$ from $f(x)$, follow these steps.
  - Replace $f(x)$ with y and interchange x and y.
  - Solve for y.
  - Replace y with $f^{-1}(x)$.

A Quick Review on Horizontal Line Test

- Any horizontal line will intersect the graph of a one-to-one function in at most one point.



Is this a one-to-one function?

- The inverse function of the one-to-one function *f* is defined as

$$f^{-1} = \{(y, x) \mid (x, y) \text{ belongs to } f\}.$$

Inverse Sine Function

- $y = \sin^{-1} x \ or \ \ y = \arcsin x$ means that $x = \sin y$, for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

By restricting the domain of the function y = sin x to the interval [-π/2,π/2] yields a one-to-one function.

By interchanging roles of x and y, we obtain the inverse sine function:

$$y = \sin^{-1} x \ or \ \ y = \arcsin x$$

Example of Finding Inverse Sine Value

- Example: Find $y = \arcsin\left(-\frac{1}{2}\right)$

Tip: Think about y as a value in radians between $-\pi/2$ and $\pi/2$, whose sine is equal to $-1/2$.

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

Remember the two special triangles.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Try to graph y = sin x and y = -1/2 in the interval $[-\pi/2, \pi/2]$, then determine the point of intersection.

Additional Examples

- $\sin^{-1}\frac{\sqrt{3}}{2}$

Since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Tip: Try to make use of one of the special triangles.

- $\sin^{-1} 2$
- Not possible to evaluate because there is no angle whose sine is 2.

Inverse Sine Function

INVERSE SINE FUNCTION
$y = \sin^{-1} x$ or $y = \arcsin x$

Domain: $[-1, 1]$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

x	y
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
0	0
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	$\frac{\pi}{2}$

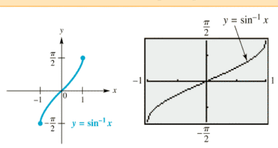


Figure 11

- The inverse sine function is increasing and continuous on its domain $[-1, 1]$.
- Its x-intercept is 0, and its y-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.

Inverse Cosine Function

By restricting the domain of the function y = cos x to the interval [0,π] yields a one-to-one function. By interchanging roles of x and y, we obtain the inverse cosine function:

- $y = \cos^{-1} x \ or \ \ y = \arccos x$ means that $x = \cos y$, for $0 \le y \le \pi$.

Tip: Think about y as a value in radians between 0 and π.

Remember the two special triangles.

- Example: Find $\arccos \frac{\sqrt{2}}{2}$

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

Inverse Cosine Function

INVERSE COSINE FUNCTION
$y = \cos^{-1} x$ or $y = \arccos x$

Domain: $[-1, 1]$ Range: $[0, \pi]$

x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0

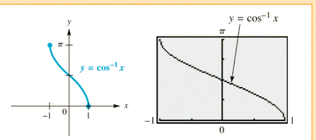


Figure 14

- The inverse cosine function is decreasing and continuous on its domain $[-1, 1]$.
- Its x-intercept is 1, and its y-intercept is $\frac{\pi}{2}$.
- Its graph is not symmetric with respect to the y-axis or the origin.

What is an Inverse Tangent Function?

- $y = \tan^{-1} x \ or \ \ y = \arctan x$ means that $x = \tan y$, for $-\pi/2 < y < \pi/2$.

Think about y as a value in radians between -π/2 and π/2.

By restricting the domain of the function y = tan x to the open interval (-π/2,π/2) yields a one-to-one function.

By interchanging roles of x and y, we obtain the inverse tangent function:

$$y = \tan^{-1} x \ or \ \ y = \arctan x$$

Inverse Tangent Function

INVERSE TANGENT FUNCTION
$y = \tan^{-1} x$ or $y = \arctan x$

Domain: $(-\infty, \infty)$ Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

x	y
-1	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$
0	0
$\frac{\sqrt{3}}{3}$	$\frac{\pi}{6}$
1	$\frac{\pi}{4}$

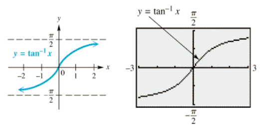


Figure 17

- The inverse tangent function is increasing and continuous on its domain $(-\infty, \infty)$.
- Its x-intercept is 0, and its y-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.
- The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

Domain and Range of Inverse Functions

Inverse Function	Domain	Range: Interval	Range: Quadrants of the Unit Circle
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I and IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I and II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I and IV
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I and II
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi], y \neq \frac{\pi}{2}$*	I and II
$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$*	I and IV

How to Find Inverse Function Values?

- Find the *degree measure* of θ in the following.
- a) θ = arctan(1) b) θ = sec^{-1}(2)

- a) θ must be in (–90°, 90°), since 1 > 0, θ must be in quadrant I. The alternative statement, tan θ = 1, leads to θ = 45°.
- b) sec(θ) = 2, for sec^{-1}(*x*), θ is in quadrant I or II. Because 2 is positive, θ is in quadrant I and sec 60° = 2. Thus, θ = 60°.

Examples

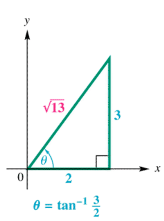
- Find y in radians if $y = \arctan(-6.24)$.
  - Set calculator in radian mode
  - Enter $\tan^{-1}(-6.24)$ $y \approx -1.411891065$
- Find y in radians if $y = \arccos 2$.
  - Set calculator in radian mode
  - Enter $\cos^{-1}(2)$ (error message since the domain of the inverse cosine function is $[-1, 1]$.

How to Find Function Values Using Definitions of the Trigonometric Functions?

- Evaluate $\sin\left(\tan^{-1}\frac{3}{2}\right)$
- Let $\theta = \tan^{-1}\frac{3}{2}$, so $\tan\theta = \frac{3}{2}$
- The inverse tangent function yields values only in quadrants I and IV, since 3/2 is positive, θ is in quadrant I.

How to Find Function Values Using Definitions of the Trigonometric Functions? (Cont.)

- Sketch and label the triangle.
- The hypotenuse is $\sqrt{13}$
- $\sin\left(\tan^{-1}\frac{3}{2}\right) = \sin\theta = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$



Another Example

- Evaluate the expression $\tan\left(2\arcsin\frac{2}{5}\right)$ without using a calculator.
- Let arcsin (2/5) = *B*

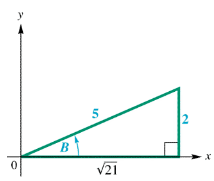
$$\tan\left(2\arcsin\frac{2}{5}\right) = \tan 2B = \frac{2\tan B}{1-\tan^2 B}$$

- Since arcsin (2/5) = *B*, sin *B* = 2/5. Sketch a triangle in quadrant I, find the length of the third side, then find tan(*B*).

Another Example (Cont.)

$$\tan\left(2\arcsin\frac{2}{5}\right) = \frac{2\left(\frac{2}{\sqrt{21}}\right)}{1-\left(\frac{2}{\sqrt{21}}\right)^2}$$

$$= \frac{\frac{4}{\sqrt{21}}}{1-\frac{4}{21}} = \frac{4\sqrt{21}}{17}$$



How to Solve a Trigonometric Equation?

- Step 1: Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
- Step 2: If only one trigonometric function is present, first solve the equation for that function.
- Step 3: If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.

How to Solve a Trigonometric Equation? (Cont.)

- Step 4: If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
- Step 5: Try using identities to change the form of the equation. If may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

Example of Solving Trigonometric Equation Using the Linear Method

- Solve 2 $\cos^2 x$ – 1 = 0
- Solution: First, solve for $\cos x$ on the unit circle.

$$2\cos^2 x - 1 = 0$$
$$2\cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm\sqrt{\frac{1}{2}}$$
$$\cos x = \pm\frac{\sqrt{2}}{2}$$

$$x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$
$$or\ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Example of Solving Trigonometric Equation by Factoring

- Solve 2 $\cos x$ + $\sec x$ = 0
- Solution:

$$2\cos x + \frac{1}{\cos x} = 0$$
$$\frac{1}{\cos x}\left(2\cos^2 x + 1\right) = 0$$
$$\frac{1}{\cos x} = 0$$
$$\frac{1}{\cos x} \neq 0$$

$$2\cos^2 x + 1 = 0$$
$$2\cos^2 x = -1$$
$$\cos^2 x = \frac{-1}{2}$$
$$\cos x = \pm\sqrt{-\frac{1}{2}}$$

$$\cos x \neq \pm\sqrt{\frac{-1}{2}}$$

Since neither factor of the equation can equal zero, the equation has no solution.

Example of Solving Trigonometric Equation by Squaring

- Solve cos x + 1 = sin x [0, 2π]

$$\cos x + 1 = \sin x$$
$$\cos^2 x + 2\cos x + 1 = \sin^2 x$$
$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x$$
$$2\cos^2 x + 2\cos x = 0$$
$$2\cos x(\cos x + 1) = 0$$
$$2\cos x = 0 \quad \text{and} \quad \cos x + 1 = 0$$
$$\cos x = 0 \qquad \cos x = -1$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \pi$$

Check the solutions in the original equation. The only solutions are $\pi/2$ and π.

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Example of Solving a Trigonometric Equation Using a Half-Angle Identity

- Solve $2\sin\frac{x}{2} = 1$
- a) over the interval [0, 2π), and
- b) give all solutions
- Solution:
  Write the interval as the inequality $0 \le x \le 2\pi$.

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Example of Solving a Trigonometric Equation Using a Half-Angle Identity (Cont.)

- The corresponding interval for x/2 is $0 \le \frac{x}{2} < \pi$.
- Solve $2\sin\frac{x}{2} = 1$

$$\sin\frac{x}{2} = \frac{1}{2}$$

- Sine values that corresponds to 1/2 are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

$$\frac{x}{2} = \frac{\pi}{6} \quad \text{or} \quad \frac{x}{2} = \frac{5\pi}{6}$$
$$x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$

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Example of Solving a Trigonometric Equation Using a Half-Angle Identity (Cont.)

- b) Sine function with a period of 4π, all solutions are given by the expressions

$$\frac{\pi}{3} + 4n\pi \text{ and } \frac{5\pi}{3} + 4n\pi$$

where *n* is any integer.

Example of Solving a Trigonometric Equation Using a Double-Angle Identity

- Solve cos (2*x*) = cos (*x*) over the interval $[0, 2\pi)$.
- First, change cos (2*x*) to a trigonometric function of *x*. Use the identity $\cos 2x = 2\cos^2 x - 1$.

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 = \cos x$$

Example of Solving a Trigonometric Equation Using a Double-Angle Identity (Cont.)

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 = \cos x$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0$$
$$\cos x = -\frac{1}{2} \quad \text{or} \quad \cos x = 1$$

- Over the interval $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$ or $x = 0$.

Example of Using the Multiple-Angle Identity

- Solve $4\sin\theta\cos\theta = \sqrt{3}$ over the interval [0, 360°).

$$2\sin\theta\cos\theta = \sin 2\theta$$
$$4\sin\theta\cos\theta = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = \sqrt{3}$$
$$2\sin 2\theta = \sqrt{3}$$
$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

Example of Using the Multiple-Angle Identity(Cont.)

- List all solutions in the interval.

$$2\theta = 60^\circ, 120^\circ, 420^\circ, 480^\circ$$
$$or\ \theta = 30^\circ, 60^\circ, 210^\circ, 240^\circ$$

- The final two solutions were found by adding 360 ° to 60° and 120°, respectively, giving the solution set $\{30^\circ, 60^\circ, 210^\circ, 240^\circ\}$.

Another Example of Using the Multiple-Angle Identity

- Solve tan 3x + sec 3x = 2 over the interval [0, 2π).
- Tangent and secant are related so use the identity

$$1 + \tan^2\theta = \sec^2\theta.$$

$$\tan 3x + \sec 3x = 2$$
$$\tan 3x = 2 - \sec 3x$$
$$\tan^2 3x = 4 - 4\sec 3x + \sec^2 3x$$
$$\sec^2 3x - 1 = 4 - 4\sec 3x + \sec^2 3x$$

Another Example of Using the Multiple-Angle Identity(Cont.)

- $\sec^2 3x - 1 = 4 - 4\sec 3x + \sec^2 3x$

$$4\sec 3x = 5$$

$$\sec 3x = \frac{5}{4}$$

$$\frac{1}{\cos 3x} = \frac{5}{4}$$

$$\cos 3x = \frac{4}{5}$$

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Another Example of Using the Multiple-Angle Identity(Cont.)

- Use a calculator and the fact that cosine is positive in quadrants I and IV,

$$3x \approx .6435, 5.6397, 6.9267, 11.9229, 13.2099, 18.2061$$

$$x \approx .2145, 1.8799, 2.3089, 3.9743, 4.4033, 6.0687.$$

- Since both sides of the equation were squared, each proposed solution must be checked. The solution set is {.2145, 2.3089, 4.4033}.

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How to Solve for *x* in Terms of *y* Using Inverse Function?

Example: $y = 3 \cos 2x$ for x.

Solution:

We want $2x$ alone on one side of the equation so we can solve for $2x$, and then for x.

$$y = 3\cos 2x$$

$$\frac{y}{3} = \cos 2x$$

$$2x = \arccos \frac{y}{3}$$

$$x = \frac{1}{2}\arccos \frac{y}{3}$$

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How to Solve an Equation Involving an Inverse Trigonometric Function?

Example: Solve 2 arcsin $x = \pi$.

Solution: First solve for arcsin x, and then for x.

$$2\arcsin x = \pi$$
$$\arcsin x = \frac{\pi}{2}$$
$$x = \sin\frac{\pi}{2}$$
$$x = 1$$

The solution set is {1}.

Another Example

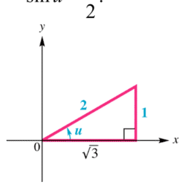
Example: Solve $\cos^{-1} x = \sin^{-1}\frac{1}{2}$.

Solution: Let $\sin^{-1}\frac{1}{2} = u$. Then sin $u = \frac{1}{2}$ and for u in quadrant I, the equation becomes

$$\cos^{-1} x = u$$
$$\cos u = x.$$

Example of Simplifying Expression Using the Half-Angle Identities

Sketch a triangle and label it using the facts that u is in quadrant I and $\sin u = \frac{1}{2}$.



Since $x = \cos u$, $x = \frac{\sqrt{3}}{2}$, and the solution set is { }.

How to Solve an Inverse Trigonometric Equation Using an Identity?

Example: Solve $\arcsin x - \arccos x = \frac{\pi}{6}$.

Solution: Isolate one inverse function on one side of the equation.

$$\arcsin x - \arccos x = \frac{\pi}{6}$$

$$\arcsin x = \arccos x + \frac{\pi}{6} \qquad (1)$$

$$\sin\left(\arccos x + \frac{\pi}{6}\right) = x$$

How to Solve an Inverse Trigonometric Equation Using an Identity? (Cont.)

Let u = arccos x, so $0 \le u \le \pi$ by definition.

$$\sin\left(u + \frac{\pi}{6}\right) = x \qquad (2)$$

$$\sin\left(u + \frac{\pi}{6}\right) = \sin u \cos\frac{\pi}{6} + \cos u \sin\frac{\pi}{6}$$

Substitute this result into equation (2) to get

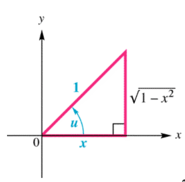
$$\sin u \cos\frac{\pi}{6} + \cos u \sin\frac{\pi}{6} = x. \qquad (3)$$

How to Solve an Inverse Trigonometric Equation Using an Identity? (Cont.)

From equation (1) and by the definition of the arcsine function,

$$-\frac{\pi}{2} \le \arccos x + \frac{\pi}{6} \le \frac{\pi}{2}$$

$$-\frac{2\pi}{3} \le \arccos x \le \frac{\pi}{3}.$$



Since $0 \le \arccos x \le \pi$, we must have $0 \le \arccos x \le \frac{\pi}{3}$.

Thus $x > 0$. From this triangle we find that $\sin u = \sqrt{1-x^2}$.

How to Solve an Inverse Trigonometric Equation Using an Identity? (Cont.)

Now substituting into equation (3) using

$$\sin u = \sqrt{1-x^2},\ \sin\frac{\pi}{6} = \frac{1}{2},$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\ \text{and}\ \cos u = x.$$

$$\sin u \cos\frac{\pi}{6} + \cos u \sin\frac{\pi}{6} = x$$

$$\left(\sqrt{1-x^2}\right)\frac{\sqrt{3}}{2} + x \cdot \frac{1}{2} = x$$

$$\left(\sqrt{1-x^2}\right)\sqrt{3} + x = 2x$$

$$\left(\sqrt{3}\right)\sqrt{1-x^2} = x$$

$$3\left(1-x^2\right) = x^2$$

$$3 - 3x^2 = x^2$$

$$3 = 4x^2$$

$$x = \sqrt{\frac{3}{4}}$$

$$x = \frac{\sqrt{3}}{2}$$

The solution set is $\{\frac{\sqrt{3}}{2}\}$.

What have we learned?

We have learned to:

1. Identify inverse functions.
2. Define and use the inverse sine function.
3. Define and use the inverse cosine function.
4. Define and use the inverse tangent function.
5. Find inverse function values.
6. Determine the solutions of a trigonometric equation within a given interval.
7. Use the inverse trigonometric functions to solve trigonometric equations.
8. Solve trigonometric equations that arise from applied problems.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Margaret L. Lial, John Hornsby, David I. Schneider, Trigonometry, 8th Edition