MAC 1114

Module 8

Applications of Trigonometry

Learning Objectives

Upon completing this module, you should be able to:

1. Solve an oblique triangle using the Law of Sines.
2. Solve an oblique triangle using the Law of Cosines.
3. Find area of triangles.

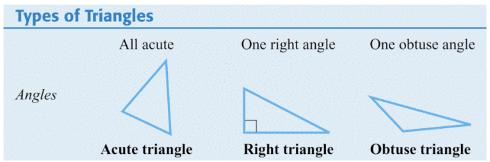
Applications of Trigonometry

There are two major topics in this module:

- Oblique Triangles and the Law of Sines
- The Law of Cosines

The law of sines and the law of cosines are commonly used to solve triangles without right angles, which is known as oblique triangles.

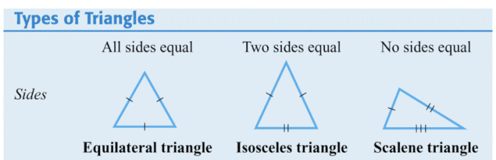
A Quick Review on Types of Triangles: Angles



How many triangles above are oblique triangles?

- Recall: The sum of the measures of the angles of any triangle is 180°.

A Quick Review on Types of Triangles: Sides



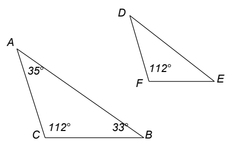
- Recall: In any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.

A Quick Review on the Conditions for Similar Triangles

- Corresponding angles must have the same measure.
- Corresponding sides must be proportional. (That is, their ratios must be equal.)

A Quick Review on Finding Angle Measures

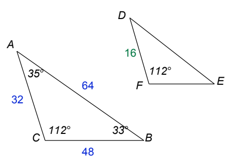
- Triangles *ABC* and *DEF* are similar. Find the measures of angles *D* and *E*.



- Since the triangles are similar, corresponding angles have the same measure.
- Angle D corresponds to angle A which = 35°
- Angle E corresponds to angle B which = 33°

A Quick Review on Finding Side Lengths

- Triangles *ABC* and *DEF* are similar. Find the lengths of the unknown sides in triangle DEF.



- To find side *DE*.

$$\frac{32}{16} = \frac{64}{x}$$
$$32x = 1024$$
$$x = 32$$

- To find side *FE*.

$$\frac{32}{16} = \frac{48}{x}$$
$$32x = 768$$
$$x = 24$$

A Quick Look at Congruency and Oblique Triangles

Side-Angle-Side (SAS)	If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.
Angle-Side-Angle (ASA)	If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.
Side-Side-Side (SSS)	If three sides of one triangle are equal, respectively, to three sides of a second triangle, then the triangles are congruent.

What Data are Required for Solving Oblique Triangles?

There are four different cases:

Case 1 One side and two angles are known (SAA or ASA)

Case 2 Two sides and one angle not included between the two sides are known (SSA). This case may lead to more than one triangle.

Case 3 Two sides and the angle included between two sides are known (SAS).

Case 4 Three sides are known (SSS).

Law of Sines

- In any triangle *ABC*, with sides *a, b,* and *c*:

$$\frac{a}{\sin A}=\frac{b}{\sin B},\quad \frac{a}{\sin A}=\frac{c}{\sin C},\quad \text{and}\quad \frac{b}{\sin B}=\frac{c}{\sin C}.$$

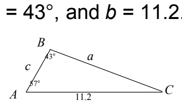
- This can be written in compact form as

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

- Note that side *a* is opposite to angle *A*, side *b* is opposite to angle *B*, and side *c* is opposite to angle *C*.

Example of Using the Law of Sines (SAA)

- In triangle *ABC*, *A* = 57°, *B* = 43°, and *b* = 11.2. Solve the triangle.
- *C* = (180 – (57 + 43))
- *C* = 180 – 100 = 80
- Find *a* and *c* by using the Law of Sines:



$$\frac{a}{\sin 57}=\frac{11.2}{\sin 43}\qquad a=\frac{11.2\sin 57}{\sin 43}\qquad a\approx 13.8$$

$$\frac{c}{\sin 80}=\frac{11.2}{\sin 43}\qquad c=\frac{11.2\sin 80}{\sin 43}\qquad c\approx 16.2$$

Therefore,

$A=57^\circ \quad a\approx 13.8$

$B=43^\circ \quad b\approx 11.2$

$C=80^\circ \quad c\approx 16.2$

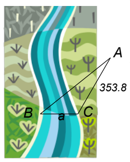
Example of Using the Law of Sines (ASA)

- Emily Miller wishes to measure the distance across High Water River. She determines that C = 110.6°, A = 32.15°, and b = 353.8 ft. Find the distance a across the river.

B = 180° − A − C

B = 180° − 32.15° − 110.6°

B = 37.25°



Example of Using the Law of Sines (SAA) Continued

- Use the law of sines involving A, B, and b to find a.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 32.15^\circ} = \frac{353.8}{\sin 37.25^\circ}$$

$$a = \frac{353.8 \sin 32.15}{\sin 37.25}$$

$$a \approx 311.04 \text{ ft}$$

Area of a Triangle (SAS)

- In any triangle ABC, the area A is given by the following formulas:

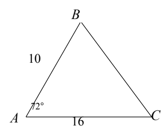
$$A = \frac{1}{2}bc\sin A, \qquad A = \frac{1}{2}ab\sin C, \qquad A = \frac{1}{2}ac\sin B.$$

- Note that side a is opposite to angle A, side b is opposite to angle B, and side c is opposite to angle C.

Example of Finding the Area: SAS

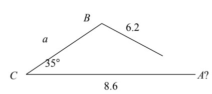
- Find the area of the triangle, *ABC* with $A = 72°$, $b = 16$ and $c = 10$.
- Solution:

$$A = \frac{1}{2}bc\sin A$$
$$A = \frac{1}{2}(16)(10)\sin 72^\circ$$
$$A \approx 76.1$$



Let's Look at One Ambiguous Case

- In triangle *ABC*, $b = 8.6$, $c = 6.2$, and $C = 35°$. Solve the triangle.



$$\frac{\sin B}{8.6} = \frac{\sin 35}{6.2}$$
$$\sin B = \frac{8.6\sin 35}{6.2}$$
$$\sin B \approx .7956$$
$$B \approx 52.7^\circ,\ 127.3^\circ$$

$$A = 180^\circ - B - C$$
$$A = 180^\circ - 52.7^\circ - 35^\circ = 92.3^\circ$$
$$or$$
$$A = 180^\circ - 127.3^\circ - 35^\circ = 17.7^\circ$$

Let's Look at One Ambiguous Case Continued

- $\dfrac{a}{\sin 92.3} = \dfrac{6.2}{\sin 35}$ *or* $\dfrac{a}{\sin 17.7} = \dfrac{6.2}{\sin 35}$

$$a = \frac{6.2\sin 92.3}{\sin 35} \qquad a = \frac{6.2\sin 17.7}{\sin 35}$$
$$a \approx 10.8 \qquad a \approx 3.3$$

- There are two solutions:

$A = 92.3^\circ$	$a \approx 10.8$		$A = 17.7^\circ$	$a \approx 3.3$
$B = 52.7^\circ$	$b \approx 8.6$	or	$B = 127.3^\circ$	$b \approx 8.6$
$C = 35^\circ$	$c \approx 6.2$		$C = 35^\circ$	$c \approx 6.2$

Number of Triangles Satisfying the Ambiguous Case (SSA)

- Let sides *a* and *b* and angle *A* be given in triangle *ABC*. (The law of sines can be used to calculate the value of sin *B*.)
  - If applying the law of sines results in an equation having $\sin B > 1$, then *no triangle* satisfies the given conditions.
  - If $\sin B = 1$, then *one triangle* satisfies the given conditions and $B = 90°$.
  - If $0 < \sin B < 1$, then either *one or two triangles* satisfy the given conditions.
    - If $\sin B = k$, then let $B_1 = \sin^{-1} k$ and use B_1 for *B* in the first triangle.
    - Let $B_2 = 180° - B_1$. If $A + B_2 < 180°$, then a second triangle exists. In this case, use B_2 for *B* in the second triangle.

Example

- Solve triangle *ABC* given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in.
- Find angle *C*.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{7.2} = \frac{\sin 43.5^\circ}{10.7}$$

$$\sin C = \frac{7.2 \sin 43.5^\circ}{10.7} \approx .46319186$$

$$C \approx 27.6^\circ$$

Example (Cont.)

- There is another angle that has sine value .46319186; it is $C = 180° - 27.6° = 152.4°$.
- However, notice in the given information that $c < a$, meaning that in the triangle, angle *C* must have measure *less* than angle *A*.
- Then $B = 180° - 27.6° - 43.5° = 108.9°$
- To find side *b*.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 108.9} = \frac{10.7}{\sin 43.5}$$

$$b = \frac{10.7 \sin 108.9}{\sin 43.5}$$

$$b \approx 14.7 \text{ in.}$$

Law of Cosines

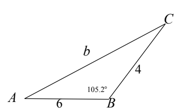
- In any triangle *ABC*, with sides *a, b,* and *c*.

$$a^2 = b^2 + c^2 - 2bc\cos A,$$
$$b^2 = a^2 + c^2 - 2ac\cos B,$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

- Thus, in any triangle, the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those sides and the cosine of the included angle between them.

Example of Using the Law of Cosines (SAS)

- Solve ΔABC if a = 4, c = 6, and B = 105.2°.



$$\frac{\sin A}{4} = \frac{\sin 105.2}{8}$$
$$\sin A = \frac{4\sin 105.2}{8}$$
$$\sin A \approx .4825$$
$$A \approx 28.8 \text{ or } 151.2$$
$$A \neq 151.2, \text{ so } A = 28.8^\circ$$

$$b^2 = 6^2 + 4^2 - 2(6)(4)\cos 105.2^\circ$$
$$b^2 \approx 64.585$$
$$b \approx 8.0$$

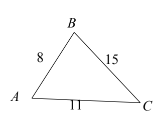
Therefore,

$A = 28.8^\circ$	$B = 105.2^\circ$	$C = 46^\circ$
$a = 4$	$b = 8.0$	$c = 6$

Example of Using the Law of Cosines (SSS)

- Solve ΔABC if a = 15, b = 11, and c = 8.
- Solve for A first

$$15^2 = 11^2 + 8^2 - 2(11)(8)\cos A$$
$$\cos A = \frac{11^2 + 8^2 - 15^2}{2(11)(8)}$$
$$\cos A \approx -.227$$
$$A \approx 103.1^\circ$$



Example of Using the Law of Cosines (SSS) Continued

- $\frac{\sin B}{11} = \frac{\sin 103.1}{15}$

  $\sin B = \frac{11 \sin 103.1}{15}$

  $\sin B \approx .7142$

  $b \approx 45.6^\circ$

$C = 180^\circ - A - B$

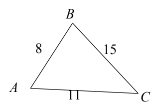
$C = 180^\circ - 102.3^\circ - 45.6^\circ$

$C = 32.1^\circ$

$A = 103.1^\circ \quad a = 15$

$B = 45.6^\circ \quad b = 11$

$C = 32.1^\circ \quad c = 8$



Summary of Possible Triangles

Oblique Triangle	Suggested Procedure for Solving
Case 1: One side and two angles are known. (SAA or ASA)	*Step 1* Find the remaining angle using the angle sum formula ($A + B + C = 180°$). *Step 2* Find the remaining sides using the law of sines.
Case 2: Two sides and one angle (not included between the two sides) are known. (SSA)	*This is the ambiguous case; there may be no triangle, one triangle, or two triangles.* *Step 1* Find an angle using the law of sines. *Step 2* Find the remaining angle using the angle sum formula. *Step 3* Find the remaining side using the law of sines. *If two triangles exist, repeat Steps 2 and 3.*

Summary of Possible Triangles Continued

Oblique Triangle	Suggested Procedure for Solving
Case 3: Two sides and the included angle are known. (SAS)	*Step 1* Find the third side using the law of cosines. *Step 2* Find the smaller of the two remaining angles using the law of sines. *Step 3* Find the remaining angle using the angle sum formula.
Case 4: Three sides are known. (SSS)	*Step 1* Find the largest angle using the law of cosines. *Step 2* Find either remaining angle using the law of sines. *Step 3* Find the remaining angle using the angle sum formula.

Heron's Area Formula

- If a triangle has sides of lengths *a*, *b*, and *c*, with semiperimeter

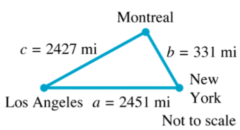
$$s = \frac{1}{2}(a + b + c),$$

- then the area of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Example of Using Heron's Area Formula

- The distance "as the crow flies" from Los Angeles to New York is 2451 mi, from New York to Montreal is 331 mi, and from Montreal to Los Angeles is 2427 mi. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)



Example of Using Heron's Area Formula (Cont.)

- The semiperimeter is

$$s = \frac{1}{2}(2451 + 331 + 2427) = 2604.5$$

- Using Heron's formula, the area *A* is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$A = \sqrt{2604.5(2604.5 - 2451)(2604.5 - 331)(2604.5 - 2427)}$$
$$A \approx 401{,}700 \text{ mi}^2$$

What have we learned?

We have learned to:

1. Solve an oblique triangle using the Law of Sines.
2. Solve an oblique triangle using the Law of Cosines.
3. Find area of triangles.

Credit

Some of these slides have been adapted/modified in part/whole from the slides of the following textbook:

- Margaret L. Lial, John Hornsby, David I. Schneider, Trigonometry, 8th Edition